Statistical Physics IV: Non-equilibrium statistical physics
ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Written exam

1. Part A (3pt/problem)

1. Write down Ito's lemma for a function $f(x)$ where $x(t)$ is a drift-diffusion stochastic process. Write down the definition of Wiener increment $dW(t)$ in the drift-diffusion process and its Ito's calculus rules (average of dW and average of dW^2).

2. Write the expression of the thermal force acting on a particle in a viscous medium at temperature T, assuming that the drag force experienced by the same particle is given by

$$F = -\gamma m v, \tag{1}$$

where v and m is the particle velocity and mass.

3. If $\{x_i\}_{i=1,..,N}$ is a set of random variables with the same average $\langle x_i \rangle = a$ and variance $\langle (x_i - \langle x_i \rangle)^2 \rangle = b^2$, and N is a large number, find the probability distribution $P(X)$ of the average $X = \frac{1}{N} \sum_{i=1}^{N} x_i$.

4. Provide an example of probability distribution to which the central limit theorem is not applicable.

5. Write down the quantum Langevin equation for the dynamics of a harmonic oscillator with frequency ω_0 and damping rate γ coupled to a bath in thermal equilibrium. If $\hat{a}$ is the annihilation operator of the oscillator what are the two-time correlators $\langle \hat{a}_{\text{in}}(t+\tau)\hat{a}^{\dagger}_{\text{in}}(t) \rangle$ and $\langle \hat{a}^{\dagger}_{\text{in}}(t+\tau)\hat{a}_{\text{in}}(t) \rangle$ for the input noise operators?

6. Write down quantum regression theorem and apply it to find correlation functions $\langle \hat{a}(t+\tau)\hat{a}(t) \rangle$ and $\langle \hat{a}(t+\tau)\hat{a}^{\dagger}(t) \rangle$ of an oscillator in thermal equilibrium. Here $\hat{a}$ is the annihilation operator, harmonic oscillator frequency is ω_0, and its damping rate is γ.

7. Why is the classical spectral density of a (stable) stochastic process $S_{xx}[\omega] = \int_{-\infty}^{\infty} \langle x(t+\tau)x(t) \rangle e^{i\omega\tau} d\tau$ always symmetric with respect to frequency ($S_{xx}[\omega] = S_{xx}[-\omega]$) and its quantum counterpart $S_{xx}[\omega] = \int_{-\infty}^{\infty} \langle \hat{x}(t+\tau)\hat{x}(t) \rangle e^{i\omega\tau} d\tau$ may not be?

8. Write down the quantum optical master equation for a zero temperature bath and for a energy relaxation process of a harmonic oscillator.

2. Part B (6pt/problem)

1. Consider a one-dimensional random walk between sites $i = 0, ..., \infty$. Assume that the particle on the n-th site is subjected to two forces, one of which makes it jump randomly to one of the neighbouring sites $n-1$ or $n+1$ with total probability per unit time being α, and the other making the particle jump only in one direction, to $n-1$, with a probability per unit time of β. Find steady state probability distribution $p(n)$ to find the particle on n-th site.

2. In the Langevin formalism without the assumption of a Markovian reservoir the damping force, F_d, acting on a particle is given by

$$F_d = -m \int_{t_0}^{t} \gamma(t-t')v(t')dt'. \tag{2}$$

Write down the Langevin equation for the velocity, v, of the particle in both the time and frequency domains. Assuming the fluctuations of velocity to be stationary calculate their spectrum, $\int_0^\infty e^{-i\omega t}\langle v(t)v(0)\rangle dt$ and relate it to the environment temperature. Using this result, drive the generalized fluctuation-dissipation theorem, i.e. find the spectrum of the random thermal force, F_{th}, defined as

$$\int_{-\infty}^{+\infty}\langle F_{\text{th}}(t)F_{\text{th}}(0)\rangle e^{-i\omega t}dt \tag{3}$$

and express it using the force susceptibility, $\chi(\omega) \equiv \frac{v(\omega)}{F(\omega)}$.

3. Derive Jarzynski equality from the Crooks theorem. Crooks theorem can be expressed in the following form:

$$\frac{P_F(W)}{P_B(-W)} = \exp\left(\frac{W-\Delta G}{k_B T}\right), \tag{4}$$

where $P_F(W)$ is the probability to perform work W over forward process, $P_B(W)$ is the probability to perform work W for backward process, ΔG is the free energy change between the initial and final states, T is the temperature and k_B is Boltzmann constant.

4. Prove that any stochastic process $X(t)$ satisfying the update formula

$$X(t+dt) = X(t) + Adt + \sqrt{D}N(0,1)\sqrt{dt}, \tag{5}$$

where A and D are constants and $N(0,1)$ is a random variable with unit normal distribution, also satisfies the forward Fokker Planck equation. You may use Ito's lemma.