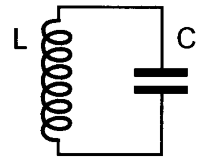
Statistical Physics IV: Non-equilibrium statistical physics
ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Exercise No.9

9.1 Quantization of an electrical LC circuit

We will quantize the excitations of a parallel inductor-capacitor (LC) circuit. A capacitor is a circuit element which stores a charge $Q(t) = CV(t)$ proportional to the voltage across its ports (C is the capacitance). The inductor is a circuit element which develops a voltage when the current $I(t)$ is changing through it: $V_L(t) = L\partial_t I(t)$ where L is the inductance. One can also define the magnetic flux $\Phi(t) = \int_0^t V(t')dt'$. The inductor stores a flux proportional to the current through it: $I(t) = \Phi(t)/L$.



(a) Write down the classical Hamiltonian (total energy) of the system in terms of the charge Q and the magnetic flux Φ.

(b) For quantization, we promote the conjugated variables Q and Φ to be quantum mechanical operators, satisfying the canonical commutation relation $[\hat{\Phi}, \hat{Q}] = i\hbar$. We can define the creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$ such that $\hat{\Phi} \propto \hat{a} + \hat{a}^\dagger$ and $\hat{Q} \propto \hat{a} - \hat{a}^\dagger$ and their commutator is $[\hat{a}, \hat{a}^\dagger] = 1$. Write down the expressions for these operators with consistent normalization and also rewrite $\hat{\Phi}$ and $\hat{Q}$ in terms of them. Express the Hamiltonian in terms of $\hat{a}$ and $\hat{a}^\dagger$.

(c) Find the zero-point vacuum fluctuations for the flux and the charge (defined as $\Delta Q_{\text{zpf}} = \sqrt{\langle 0|\hat{Q}^2|0\rangle - \langle 0|\hat{Q}|0\rangle^2}$ for the charge and similarly for the flux, with $|0\rangle$ the ground state of the harmonic oscillator) and show that they satisfy the Heisenberg uncertainty relation.

9.2 The Sudarshan-Glauber representation of a density matrix

The coherent states,

$$|\alpha\rangle = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} |0\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle ,$$

have unique properties that enable them to be used as a valid basis to represent operators in Hilbert space.

(a) Show that the set of coherent states over the entire complex $\alpha-$plane provides a resolution of the unity,

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle\alpha| = \mathbb{1};$$

here and henceforth, we denote $d^2\alpha = d(\text{Re}[\alpha])d(\text{Im}[\alpha])$

(b) Owing to the non-orthogonality of coherent states, they form a linearly dependent set. Show that in fact any coherent state can be expressed in terms of other coherent states as,

$$|\alpha\rangle = \frac{e^{-|\alpha|^2/2}}{\pi} \int e^{-|\beta|^2/2+\beta^*\alpha} |\beta\rangle \; d^2\beta.$$

Thus, the coherent states form an overcomplete basis.

(c) Now show that any operator O has the following representation in terms of coherent states,

$$\hat{O} = \int O(\alpha, \beta) |\alpha\rangle \langle\beta| \; d^2\alpha \; d^2\beta,$$

where $O(\alpha, \beta) = \pi^{-2} \langle\alpha| O |\beta\rangle$ is a complex function corresponding to the operator $\hat{O}$.

(d) However, owing to the linear dependence of coherent states, the function $O(\alpha,\beta)$ is not unique. In fact, show that the set of possible functional correspondences satisfy the relation,

$$O(\alpha,\beta)=\frac{e^{-(|\alpha|^2+|\beta|^2)/2}}{\pi^2}\int O(\alpha',\beta')e^{-(|\alpha'|^2+|\beta'|^2)/2+\alpha^*\alpha'+\beta(\beta')^*}d^2\alpha'\ d^2\beta'$$

(e) If further the operator $\hat{O}$ is hermitian, i.e $\hat{O}^\dagger=\hat{O}$, we must have $O(\alpha,\beta)^*=O(\beta,\alpha)$. To remove the redundancy arising from the overcompleteness, we insist now that $O(\alpha,\beta)$ is a real function; then the function must be a function of a single variable, and should thus furnish a diagonal representation in terms of coherent states. Thus, we must have a function P such that,

$$O=\int P(\alpha)\left|\alpha\right\rangle\left\langle\alpha\right|\ d^2\alpha.$$

Prove now that this representation is unique by inverting it:

$$\Rightarrow P(\beta)=\int O(-\alpha,\alpha)e^{|\alpha|^2+|\beta|^2+\alpha^*\beta-\beta^*\alpha}\ d^2\alpha.$$

This is the Sudarshan-Glauber P-representation[1] for hermitian operators – in particular, this is profitably employed for the density operator.

9.3 Quantum Langevin equation for a harmonic oscillator interacting with a heat bath [2]

Consider a harmonic oscillator coupled to a heat bath, consisting of a large ensemble of harmonic oscillators. The Hamiltonian of the system and the bath can be expressed as

$$\hat{H}_{\text{sys}}=\hbar\omega_s\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)\tag{1}$$

$$\hat{H}_{\text{bath}}=\sum_k\hbar\omega_k\left(\hat{b_k}^\dagger\hat{b}_k+\frac{1}{2}\right)\tag{2}$$

Here, $\hat{a}$ and $(\hat{a}^\dagger)$ are the annihilation and creation operators which satisfy the commutator relation $[\hat{a},\hat{a}^\dagger]=1$ and can be related to the position operator via $\hat{x}_{\text{sys}}=\sqrt{\frac{\hbar}{2m\Omega_m}}(\hat{a}^\dagger+\hat{a})$. Assume that the bath and system are interacting in a bilinear way, i.e. that the interaction Hamiltonian takes the form:

$$\hat{H}_{int}=\hbar\sum_k g_k(\hat{a}\hat{b}_k^\dagger+\hat{b}_k\hat{a}^\dagger)\tag{3}$$

We also assume that the heat bath is in thermal equilibrium and has a finite temperature. This implies that $\langle\hat{b}_k^\dagger(0)\rangle=\langle\hat{b}_k(0)\rangle=0$, as well as $\langle\hat{b}_k^\dagger(0)\hat{b}_l(0)\rangle=\delta_{k,l}\bar{n}_k$ and $\langle\hat{b}_k(0)\hat{b}_l^\dagger(0)\rangle=\delta_{k,l}(\bar{n}_k+1)$ where $\bar{n}_k$ is the effective occupation number of the k^{th} mode. Finally, we assume that the bath modes are initially uncorrelated with each other, i.e. $\langle\hat{b}_k(0)\hat{b}_l^\dagger(0)\rangle=0$ for $k\neq l$.

1. The interaction between two harmonic oscillators - analogously to the coupling of two classical harmonic oscillators - is given by: $\hat{H}_{int}=G\hat{x}_{\text{sys}}\hat{p}_{\text{bath}}$, where $\hat{x}_{\text{sys}}$ is the position operator of the system and $\hat{p}_{\text{bath}}$ the momentum operator of the bath mode. State precisely under which assumptions $\hat{a}^\dagger\hat{b}_k^\dagger$ and $\hat{b}_k\hat{a}$ which arise in this case, can be neglected (which is termed the *rotating wave approximation*).

2. Derive the Heisenberg equations of motion for the bath and system operators separately.

[1] E. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963); R. Glauber, Phys. Rev. **131**, 2766 (1963)

[2] See e.g. Scully, *Quantum Optics*, Chapter 9 or Gardiner and Collet, *"Input and output in damped quantum systems: Quantum stochastic differential equations and the master equation"*, Physical Review A (1984)

3. Next, eliminate the bath operators from the equation of motion for $\hat{a}(t)$ by inserting the equation of the bath. Moreover, introduce the *density of states* $D(\omega)$ (that is, the number of modes in a given volume between ω and $\omega + d\omega$) of the bath modes and convert the summation over k to an integral over ω, assuming that the coupling is frequency independent, i.e. $g_k = g(\omega_k) = g$ (this is called the 1st Markov approximation). Carry out the integration over ω using $\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} = 2\pi\delta(t-t')$.[3]

4. Transform the equations of motion to a frame rotating with ω_s with respect to the original Hamiltonian $H_0 = H_{\text{sys}} + H_{\text{bath}}$. Show that the equation of motion for the system operator obeys the **Quantum Langevin Equation**:

$$\boxed{\frac{d}{dt}\hat{a}(t) = -\frac{\kappa}{2}\hat{a}(t) + \hat{F}(t)}$$

Give an expression for the Langevin noise term $\hat{F}(t)$ in terms of the bath operators.

5. Show that the noise operator $\hat{F}(t)$ satisfies

$$\langle \hat{F}(t)^{\dagger}\hat{F}(t')\rangle = \kappa\bar{n}_{th}\delta(t-t'),$$

$\langle \hat{F}^{\dagger}(t)\rangle = 0$ and show that the system's decay rate is given by: $\kappa = 2\pi D(\omega_s)|g|^2$.

6. Show that the quantum Langevin equation - despite containing now dissipation and damping of the system operator - preserves the commutator, $[\hat{a}(t), \hat{a}^{\dagger}(t)] = 1$. We have achieved a quantum mechanically consistent description of damping.

7. Derive and solve the equations of motion for the mean field $\langle \hat{a}(t)\rangle$.

8. How does the (average) number of photons evolve in time? Calculate $\frac{d}{dt}\hat{N} = \frac{d}{dt}\left(\hat{a}^{\dagger}\hat{a}\right)$ and $\frac{d}{dt}\langle \hat{N}\rangle = \frac{d}{dt}\langle \hat{a}^{\dagger}\hat{a}\rangle$.

9.4 Quantum Langevin equation for a two-level system interacting with a harmonic oscillator heat bath(*)[4]

In this exercise, we derive the quantum Langevin equations for a two level system, coupled to a harmonic oscillator heat bath. This model can serve to describe the decay of an excited atom (to its electronic ground state) via the phenomenon of spontaneous emission. You will derive the spontaneous emission rate (using the Wigner-Weisskopf theory of spontaneous emission). The bath modes are the modes of the electromagnetic field around the atom. The Hamiltonian of the system, the bath and the interaction are given by:

$$H_{sys} = \frac{\hbar\omega_s}{2}\hat{\sigma}_z$$

$$H_{bath} = \sum_k \hbar\omega_k\left(\hat{b}_k^{\dagger}\hat{b}_k + \frac{1}{2}\right)$$

$$H_{int} = \sum_k \hbar g_k(\hat{b}_k^{\dagger}\hat{\sigma}_- + \hat{\sigma}_+\hat{b}_k)$$

Here $\{\hat{\sigma}_-, \hat{\sigma}_+, \hat{\sigma}_z\}$ are the Pauli-operators for the spin-1/2 (two level) system and g_k is the coupling coefficient between the atom and the electromagnetic field[5].

[3] Note that this property also gives rise to the equality $\int_{t_0}^{t} c(t')\delta(t-t')dt' = \frac{1}{2}c(t)$, which is needed to derive the exact form of the quantum Langevin equation (see next point).

[4] This model describes e.g. the irreversible decay of an excited atom, i.e. spontaneous emission See e.g. Scully, "Quantum Optics", Chapter 6.

[5] The coupling coefficient is given by $g_k = \frac{-\rho_{12}}{\hbar}E_{\text{zpf}}$ where ρ_{12} is the off-diagonal matrix elements of the dipole operator $\hat{p} = e\hat{r}$ and E_{zpf} is the zero point fluctuation of the electromagnetic field mode (k)

1. Repeat the adiabatic elimination procedure outlined in class and derive the quantum Langevin equations.

2. Derive the equations of motion for the expectation values of the system operators $\langle\hat{\sigma_z}\rangle$, $\langle\hat{\sigma_+}\rangle$, $\langle\hat{\sigma}_-\rangle$ and show that the equations of motion lead to a system that is not closed.
HINT: Solve the equations for the bath modes formally. Use this and the expected value to eliminate some terms, then convert the sum over k to an integral. Use the Markov approximation.

3. Use these equations to calculate the spontaneous emission rate of an atom - the rate at which it spontaneously decays from an excited state to its ground state by emitting a photon -, a seminal result in the quantum theory of light-matter interaction. For this, solve the equation of motion by considering that the bath modes are at zero temperature $\langle\hat{b_k}(t)^\dagger\hat{b_k}(t)\rangle = 0$ and by assuming that the atom is initially in the excited state, i.e. $\langle\hat{\sigma_z}(0)\rangle = 1$. Derive the spontaneous emission rate $\Gamma_{e\to g}$. Consider only a two level system!