Statistical Physics IV: Non-equilibrium statistical physics
ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Exercise No.6

6.1 Distribution of maximum step-sizes in Lévy flights[1]:

The scaling of the average maximum step sizes is an important characteristic, which determines if a random process is dominated by rare events (i.e. an anomalous diffusion). This problem considers one-dimensional discrete random walks in two cases, the case of a regular Wiener process (Brownian motion) and the case of Lévy Flights, when the probability distribution of step sizes is given by the Cauchy distribution.

(a) First derive an expression for the probability $Q(\bar{y}, n)$ to observe during n jumps there is only one step bigger than $\bar{y}$. Assume that the position increment $y = x_j - x_{j+1}$ of each step obeys the probability distribution $P(y)$ (here x_j is the particle position at the time t_j). Next find the *most probable* step size $\bar{y}_n$, and show that it satisfies the integral-equation $\int_{\bar{y}_n}^{\infty} P(y)\ \mathrm{d}y = \frac{1}{\mathrm{n}}$.

(b) Find the maximum step size distribution in the case of a *Wiener* process, i.e. $P(y) = \frac{1}{\sqrt{4\pi D\tau}} \exp\left(-\frac{y^2}{4D\tau}\right)$. Show that the most probable maximum step size in the limit of large n does not contribute significantly to the second order moment of $x(n)$, i.e. show that $\lim_{n\to\infty}[\frac{\bar{y}_n}{\langle x(n)^2\rangle^{1/2}}] = 0$ by deriving the expression for $\bar{y}_n$. Here $\langle x(n)^2\rangle$ is the second order moment of the variable x after n time lapses ($t = n\tau$).

(c) Next consider the case of a Cauchy process $P(y) \simeq \frac{b}{y^{1+\mu}}$ (for $y \to \infty$), where $\mu = 1$. Using the result that the first order moment scales as $\langle x(n)\rangle \simeq n^{1/\mu}$, show that the $lim_{n\to\infty}[\frac{\bar{y}}{\langle x(n)\rangle}]$ does not converge to zero. This implies that one rare event (i.e the most probable maximum step) can contribute as much as the average value of position and dominates the statistics. This is referred to as a *Lévy flight*.

(d) Prove that for the conditional probability distribution $P(y) \propto \frac{b}{y^{1+\mu}}$ the second order moments $\langle y^2\rangle$ diverge if $0 < \mu < 2$, while the first order moments $\langle y\rangle$ diverge if $0 < \mu < 1$.

6.2 Arrhenius cascade[2]

Consider a particle in a washboard potential, characterized by a series of potential wells of height V_i. Moreover assume that the individual potential wells are tilted, i.e. one has a tilted washboard potential (cf. Figure 3.8 Martin). Assume that the thermal fluctuations enable the particle to escape from the well with Kramer's escape rate of $r = 1/\tau_i = 1/\tau_0 exp(-V_i/k_B T)$, with τ_i the characteristic dwelling time and τ_0 a constant. The potential barriers V_i are randomly distributed, according to an exponential distribution, $P(V) = \frac{1}{E_0}\exp\left(-\frac{V}{E_0}\right)$, with E_0 the average well height.

1. Derive an expression for $\langle\tau(n)\rangle = \sum_{i=1}^{n}\langle\tau_i\rangle$ for the case of constant well heights and exponentially distributed ones. For $\mu = k_B T/E_0 < 1$, the process becomes a Lévy flight, and the average is no longer defined.

2. (*) Simulate the time $\tau(n)$ that is necessary for the particle (starting in the highest potential at $i = 1$), to escape the cascade and arrive at $i = n$. Contrast the result for $\mu < 1$ and $\mu > 1$. In particular, highlight the scaling $\tau(n) \sim n^{1/\mu}$ for $\mu < 1$.

[1]Martin: Advanced Statistical Mechanics, Chapter 3.
[2]cf. Chapter 4.1 of "Levy Statistics and Laser Cooling" Bardou, Bouchard, Aspect, Cohen-Tannoudji

6.3 Simulation of Lévy flights(*)

In this exercise we are going to simulate a Lévy flight. Consider a discrete 1D random walk that the position of the particle after n steps is given by $x(n)$. Assume that the increments y_n, defined as $y_n = x(n+1) - x(n)$, are given by $y_n = |z_n|$ where z_n is a Cauchy distributed random variable with the probability distribution

$$P(z_n) = \frac{b}{\pi(b^2 + z_n^2)}, \quad (-\infty < z_n < \infty). \tag{1}$$

Perform a simulation of the random process for different values of b [3].

1. First, show that the first moment of $x(n)$ does not converge for large n. Do so by plotting $\langle x(n) \rangle$ as a function of n.
2. Next, show that for large n, $x(n)$ has the scaling behaviour as: $\langle x(n) \rangle \simeq bn \ln(n)$ (i.e. the process results in anomalous diffusion).
3. Finally, for large steps sizes n, find via numerical simulation and drawing histograms the probability distribution of $P(u)$ with $u = \frac{x_n}{n}$. Find the power law of tail of this distribution by fitting a function of the form u^α to the tail.

[3] Note that Python, Mathematica and MatLab have built-in functions to generate numbers from a Cauchy-distribution.