Statistical Physics IV: Non-equilibrium statistical physics
ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Exercise No.4

4.1 The Generalized Fluctuation Dissipation Theorem[1]

The Generalized Fluctuation dissipation theorem as derived by Callen and Greene[2] states that the spectral density of thermal Langevin force fluctuations obey:

$$|F_L(\omega)|^2 = 4k_B T \cdot R(\omega) \tag{1}$$

where $R(\omega)$=Re$\{Z(\omega)\}$ is the mechanical *resistance* which is the real part of the *complex* valued mechanical *impedance* $Z(\omega) \equiv F_L(\omega)/v(\omega)$, where $F_L(\omega)$ is the random thermal force and $v(\omega)$ is the velocity (quantities are in the Fourier domain). Equivalently, this implies that the velocity fluctuations satisfy:

$$\boxed{S_{vv}(\omega) = 4k_B T \cdot \frac{R(\omega)}{|Z(\omega)|^2}}$$

1. Calculate the spectrum of velocity fluctuations $S_{vv}(\omega)$ of a harmonic oscillator of mass m, and velocity proportional damping γ (i.e. $F_D(t) = \gamma v(t)$). State the value of R and Z.

2. Calculate the spectrum of position fluctuations $S_{xx}(\omega)$.

3. Let us next assume a non-trivial damping force and apply the generalized FDT. Assume we study the velocity fluctuation spectrum of a mass on a spring with spring constant k. The associated damping is given by the material induced, intrinsic damping force $F_D(\omega) = ik\phi(\omega)x(\omega)$. This means that if an external sinusiodal force is applied to the oscillator, position $x(t)$ lags behind the force by angle $\phi(\omega)$, which is in general dependent on the frequency of excitation. (Zener proposed that in solids this angle obeys: $\phi(\omega) = \frac{\Delta\omega\tau}{1+(\omega\tau)^2}$where τ is some characteristic damping time intrinsic to the material). Use the generalized fluctuation dissipation theorem to find the form of the fluctuation spectrum $S_{vv}(\omega)$ and $S_{xx}(\omega)$

4.2 Stationary solutions of the Fokker Planck Equation[3]:

The one dimensional Fokker Planck equation with constant drift (Smoluchowski equation) is given by: $\frac{\partial}{\partial t}P(x,t) = -\frac{1}{\gamma}\frac{\partial}{\partial x}(F(x)P(x,t)) + D \cdot \frac{\partial^2}{\partial x^2}P(x,t)$, where D is a the diffusion constant, $F(x) = -dV(x)/dx$ the consrvative force and γ is the dissipation constant. This equation can also be written in the form of a continuity equation:

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}J(x,t)$$

Where the probability current is $J(x,t) = \left[-D \cdot \frac{\partial}{\partial x} + \frac{1}{\gamma}F(x)\right] P(x,t)$.

1. First, for a stationary probability distribution ($\frac{\partial}{\partial t}P(x,t) = 0$), J must be constant. Assume first that the probability current vanishes somewhere, this implies $J(x) = 0 \forall x$. Find that in this case the stationary probability distribution $P(x,t)$ has the form $P(x) = Ne^{-\Phi(x)}$and derive $\Phi(x)$.

[1]See "Thermal noises in mechanical experiments", Saulson, Phys. Rev. D. Vol 42, No. 8, (1990) - Section III
[2]"On a theorem of irreversible thermodynamics", Callen and Greene, Physical Review (1952)
[3]cf. Risken: "The Fokker Planck Equation", chapter 4, Springer Verlag

2. *Boundary conditions of the Fokker Planck equation*: Show that if the probability current J vanishes at the boundaries at $x = x_{min}$ and $x = x_{max}$, it follows that $\int_{x_{min}}^{x_{max}} P(x,t)dx = const$. Hence the probability distribution is constant with time (no particle can escape or can be absorbed, i.e. we have *reflecting boundaries*).

3. Next assume the case where probability current J is constant but *non-zero*. Derive the stationary probability distribution $P(x)$ for this case. Assume that the problem is solved on a finite interval $J(x_{min}) = J(x_{max}) = J$. What do the boundary conditions now imply? Explain why in this situation (despite having a non-zero current J) the distribution $P(x)$ can still be stationary.

4.3 Fokker Planck Equation to derive the limit of atomic laser cooling(*)[4]:

Atom laser cooling proceeds by exposing atoms to a standing wave laser field (i.e. a sum of right and left propagating fields in a 1D-case) whose frequency ω_L is slightly below the atomic transition frequency ω_0. From basic quantum theory of atom-photon interaction it follows, that when a photon is absorbed by or emitted from an atom there is a momentum recoil given by $\Delta p = \hbar k$ ($k = \omega_L/c$).

In the case of absorption the recoil is co-directional with the propagation of the photons being absorbed. Due to the fact that the atoms move and experience a Doppler shift, the rates $\epsilon_{\pm}^{abs}(p)$ at which the atom absorbs light from the right or left lasers are different and related as

$$\epsilon_{\pm}^{abs}(p) = \sigma_{\pm}\frac{I}{\hbar\omega_L} = s\frac{(\gamma/2)^2}{(\Delta\omega \pm kv)^2 + (\gamma/2)^2}, \tag{2}$$

where I is the laser intensity, $s = \sigma_0 I/\hbar\omega_L$ is the photon flux through the resonant scattering cross-section of the atom ($\sigma_0 = 2\pi\lambda^2$), γ is the spontaneous emission rate, $v = p/m$ speed of the atom and $\Delta\omega = \omega_L - \omega_0$. In contrast, the spontaneous emission events provide recoils in a random direction ($+$ or $-$ in 1D-case) in each direction at the rate

$$\epsilon_{\pm}^{em}(p) = \frac{\epsilon_{+}^{abs}(p) + \epsilon_{-}^{abs}(p)}{2}, \tag{3}$$

so that on average the atom emits and absorbs equal number of photons per unit time.

Over the course of the excercise assume the laser detuning $\Delta\omega = -\gamma/2$ (which turns out to lead to the lowest achievable temperature).

1. Using these expressions, derive the 1-dimensional Fokker Planck equation for the case of atomic laser cooling by considering a *one dimensional random walk in momentum space* described by the probability distribution $P(p,t)$ and give the expression for the drift and diffusion coefficients. Start your derivation by expanding $P(p, t + \Delta t)$, similar to the derivation of the Smoluchowski equation. Assume here the limit $kv \ll \gamma$ and use the following formulas

$$\epsilon_{\pm} = \epsilon_{\pm}^{\text{abs}} + \epsilon_{\pm}^{\text{em}},$$

$$(\epsilon_{+}(p) - \epsilon_{-}(p))\Delta p \approx -\beta v,$$

$$(\epsilon_{+}(p) + \epsilon_{-}(p))(\Delta p)^2 = 2D,$$

where $\beta = sk^2\hbar/(\gamma/2)$ and $D = sk^2\hbar^2$.

[4]cf. Physics Nobel Prize 1997 to Steven Chu, Claude Cohen-Tannoudji and William D. Phillips, For development of methods to cool and trap atoms with laser light. For a dervation cf: S. Stenholm, The Semiclassical Theory of Laser Cooling, Rev. Mod. Phys. 1986, Chapter 5

2. Solve the steady state ($\frac{\partial}{\partial t}P(p,t) = 0$) of the Fokker Planck equation and show that the distribution has the form $P_{ss}(p,t) \propto e^{-p^2/(2mk_B T_{\rm eff})}$, where the effective temperature $T_{\rm eff}$ is defined so that $k_B T_{\rm eff} = \hbar\gamma/2$. This is the famous *Doppler limit* of atomic laser cooling[5].[6]

3. Return to the more general case where the cooling force only acts over a finite range of velocities (i.e. physically this comes from the fact that if the atoms travel too fast, their Doppler shift can exceed their linewidth implying that the cooling laser is entirely off resonant and will thus not excite the atoms). Then kv may be $> \gamma$ and more general expressions for the momentum jumps rates should be used, so that

$$(\epsilon_+(p) - \epsilon_-(p))\Delta p = \frac{-\beta \cdot v}{1 + (v/v_c)^4},$$

$$(\epsilon_+(p) + \epsilon_-(p))(\Delta p)^2 = 2D\frac{1 + (v/v_c)^2}{1 + (v/v_c)^4},$$

where $v_c = \gamma/(k\sqrt{2})$ is the "capture velocity". Derive for this case the steady-state momentum probability distribution $P(p)$.

4.4 Spectral broadening of a laser by phase diffusion[7]

We consider an oscillating electromagnetic field that exhibits phase fluctuations, which model, for example, the output of a realistic RF signal generator or a laser. Assume that the field is given by $E(t) = E_0 e^{-i\omega_0 t + i\phi(t)}$, where the phase $\phi(t)$ is a Wiener process with diffusion constant D.

1. Show that the phase fluctuations satisfy

$$\left\langle e^{i\phi(t_1) - i\phi(t_2)} \right\rangle = e^{-D|t_1 - t_2|}.$$

2. Show that this leads to the electric field spectrum $S_{EE}(\omega) = \int_{-\infty}^{\infty} \langle E(t)E^*(t+\tau)\rangle\, e^{i\omega t} d\tau$ of Lorentzian shape. Relate the linewidth of the Lorentzian to the diffusion constant D.

3. In many practical situations phase fluctuations are described not by a simple Wiener process, which results in the constant power spectral density of frequency fluctuations (white noise), but rather by a stochastic process with frequency noise increasing at low frequencies. In this case the electric field spectrum is not necessarily a Lorenzian, and moreover its linewidth might depend on the observation time $T_{\rm obs}$. For a common model example [8] when the frequency noise spectral density is given by $S_\nu = (\omega/2\pi)^2 S_\phi = k/|\omega|$, the phase diffusion approximately obeys

$$\langle (\phi(t_1) - \phi(t_2))^2 \rangle = (t_1 - t_2)^2 \frac{k}{\pi} \left(a + \log\left(\frac{a T_{\rm obs}^2 k}{\pi} \right) \right),$$

where $a = 4.3$ is a numeric constant.

Calculate the shape of electric field spectrum in this case.

[5]The 1997 Nobel prize was in particularly awarded for methods to cool below this Doppler limit (techniques called sub-Doppler laser cooling).

[6]The Doppler limit can be viewed as a manifestation of the Heisenberg uncertainty limit. A photon that cools the atom, decays within a time frame of $\Delta t = \gamma^{-1}$. As a result the atom has an energy uncertainty given by $\Delta E \cdot \Delta t > \hbar/2$. Hence $\Delta E > \hbar\gamma/2$.

[7]See, for example, Riehle F. "Frequency Standards: Basics and Applications", section 3.4

[8]Mercer L. B. "1/f frequency noise effects on self-heterodyne linewidth measurements". Journal of Lightwave Technology 9, 485–493 (1991).