Statistical Physics IV: Non-equilibrium statistical physics
ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Exercise No.2

2.1 Stochastic differential equations

A *continuous, memoryless, stochastic* [1] function of time $X(t)$ satisfies the following update formula for the increment in time dt:

$$X(t+dt) = X(t) + A(X(t),t)dt + \sqrt{D(X(t),t)}N(0,1)\sqrt{dt}, \tag{1}$$

where $A(x,t)$ is the drift, $D(x,t)$ is the diffusion function (both continuous in their arguments), and $N(0,1)$ is a random variable with normal distribution of unit variance.

(a) Show that the update formula without N being a random variable cannot describe a continuous memoryless process, due to violation of self-consistency.
Hint: Examine the update induced by $t \to t+dt$; and compare it with the result obtained by performing the same update in two steps, $t \to t+dt/2 \to t+dt$.

(b) Show that Eq. 1 does satisfy the self-consistency requirement.

(c) Derive the equation of motion for the mean $\langle X(t+dt)\rangle$ and the second moment $\langle X(t+dt)^2\rangle$ by directly applying Eq. 1. Express the results as ordinary differential equations.

(d) Show that Eq. 1 is equivalent to the white noise Langevin equation:

$$\frac{dX}{dt} = A(X(t),t) + \sqrt{D(X(t),t)}\Gamma(t). \tag{2}$$

What is the explicit form of $\Gamma(t)$; and what are the average and the two-point correlators of Γ?

2.2 Wiener-Khinchin theorem

For a stochastic process $X(t)$, prove that the Fourier transform of the auto-correlation

$$C_{XX}(\tau) = \langle X(t)X(t+\tau)\rangle$$

is related to the power spectrum $S_{XX}(\omega)$ via

$$C_{XX}(\tau) = \int S_{XX}(\omega)e^{-i\omega\tau}\frac{d\omega}{2\pi}$$

Where the spectrum density is defined as $S_{XX}(\omega) = \langle|\tilde{X}(\omega)|^2\rangle$, with the normalized Fourier transform $\tilde{X}(\omega) = \lim_{T\to\infty}\frac{1}{\sqrt{T}}\int_{-T/2}^{T/2}X(t)e^{i\omega t}\,dt$.

2.3 Review of Spectral Densities

Spectral densities are an often encountered and essential concept in both Engineering and Physics. However, their definition can differ as they can be one- or two-sided spectral densities. This exercise is intended to clarify the difference and use of both notations. In Physics the definition of spectral densities one often encounters is two-sided (extending over both negative and positive Fourier frequencies): $S_{XX}(\omega) = \langle|\tilde{X}(\omega)|^2\rangle = \lim_{T\to\infty}\langle|\frac{1}{2\pi}\int_{-T/2}^{T/2}X(t)e^{i\omega t}dt|^2\rangle$. The Wiener-Khinchin theorem (see next problem) then implies that $S_{XX}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}C_{XX}(\tau)e^{i\omega\tau}d\tau$ where $C_{XX}(\tau) = \langle X(t)X(t+\tau)\rangle$.

[1] cf. D.T. Gillespie, "The mathematics of Brownian motion and Johnson noise", Am. J. Phys. **64**, 225 (1996)

(a) Show that for the Orstein-Uhlenbeck process (defined with drift $A(X,t) = -1/\tau \cdot X$ and diffusion $D(X,t) = c$) the auto-correlation function $C_{XX}(\tau) = \langle X(t)X(t+\tau)\rangle$ is symmetric[2], i.e. $C_{XX}(\tau) = C_{XX}(-\tau)$.

(b) Show that for a symmetric autocorrelation function $C(\tau) = C(-\tau)$ the *two-sided spectrum* $S_{XX}(\omega)$, as defined by $S_{XX}(\omega) = \int C(\tau)e^{-i\omega\tau}d\tau$, is real, non-negative and even[3], i.e. $S_{XX}(\omega) = S_{XX}(-\omega)$. This - and the fact that in experimental science negative frequencies cannot be readily measured - motivates the use of a *one-sided* spectral density $S_{XX}^{\text{single}}(\omega) = 2 \cdot S_{XX}(\omega)$ where, $\omega \geq 0$.

(c) Show that the definitions of a double sided spectral density $S_{XX}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\tau)e^{i\omega\tau}d\tau$ and $S_{XX}^{\text{single}}(\omega) = 4\frac{1}{2\pi}\int_0^{\infty} C(\tau)cos(\omega t)d\tau$ lead to $S_{XX}^{\text{single}}(\omega) = 2S_{XX}(\omega)$, where $\omega \geq 0$.

(d) Returning to the Ornstein-Uhlenbeck process ($A(X,t) = -1/\tau \cdot X$ and $D(X,t) = c$) calculate the two-sided spectral density of fluctuations $S_{XX}(\omega)$ as well as the one-sided spectral density, $S_{XX}^{\text{single}}(\omega)$.

2.4 Johnson noise and its spectrum(*)

Johnson noise (discovered by J.B. Johnson, and explained by H. Nyquist) refers to the random fluctuations of the voltage across a resistor, due to the thermal motion of charge carriers.

(a) Consider a resistance R in series with an inductor L, for which Khirchoff's laws imply

$$-RI(t) - L\frac{dI(t)}{dt} + V(t) = 0.$$

Here, the voltage V is a random stochastic function. Bring this equation into the form of a standard O-U process and determine the constants c and τ by supposing that the system reaches thermal equilibrium.

(b) From this derive the fluctuation-dissipation relation

$$\langle V(t)V(t')\rangle = 2k_BTR\delta(t-t').$$

(c) Using the Wiener-Khinchin theorem, calculate the spectrum of voltage fluctuations $S_{VV}(\omega)$. The result $S_{VV}(\omega) = 2k_BTR$ is the Nyquist formula[4]. The frequency-independence of the noise spectrum motivates the term *white noise*.

[2]Note that because $X(t)$ is a real-valued and a classical variable, $C_{XX}(\tau)$ is always real and symmetric in time since $\langle X(t)X(t')\rangle = \langle X(t')X(t)\rangle$. This is not the case for a quantum mechanical operator. (See e.g. "Introduction to Quantum Noise and Measurement" Clerk et al. Rev. Mod. Phys.).

[3]It is important to note that in Quantum Physics, the use of a two-sided spectral density is essential, as the spectral densities are not generally symmetric - due to the non-zero commutation relations - as shown later in this class. See e.g. "Introduction to Quantum Noise and Measurement" Clerk et al. Rev. Mod. Phys.

[4]Note that in the literature, the formula is often given for (experimentally more relevant) one-sided power densities (i.e. for positive frequencies only), so that it reads $4k_BTR$.