Statistical Physics IV: Non-equilibrium statistical physics
ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Exercise No. 11

11.1 Quantum master equation for a damped harmonic oscillator[1]

The master equation describing the evolution of the reduced density matrix $\hat{\rho} = \mathrm{Tr}_B\,\hat{\rho}_{SB}$ (traced over bath variables), takes the Lindblad form

$$\frac{d\hat{\rho}}{dt} = \frac{-i}{\hbar}[\hat{H}_0, \hat{\rho}] + \sum_k \left(\hat{L}_k \hat{\rho} \hat{L}_k^\dagger - \frac{1}{2}\{\hat{L}_k^\dagger \hat{L}_k, \hat{\rho}\} \right),$$

where $\hat{H}_0 = \hbar\omega \hat{a}^\dagger \hat{a}$ and where the $\hat{L}_k$ are the Lindblad jump operators describing each of the many possible decoherence mechanisms influenced by the bath degrees of freedom.

Here we consider a harmonic oscillator coupled to a thermal bath such that the bath affects the following two physical decoherence mechanisms:

- *Energy relaxation.* In this case the oscillator loses quanta into the bath, a process described by the Lindblad operator $\hat{L}_{1,-} = \sqrt{\gamma_1(\bar{n}+1)}\,\hat{a}$. This process is invariably accompanied by the stimulation of excitations in the oscillator by the bath, described by the jump operator $L_{1,+} = \sqrt{\gamma_1 \bar{n}}\, a^\dagger$. Here $\gamma_1 = 1/T_1$ is the energy relaxation rate of the oscillator, and $\bar{n}$ its Bose-Einstein occupation at given bath temperature.

- *Phase damping.* In this case, there is no exchange of quanta between the bath and the oscillator, however the bath affects the oscillator by *resetting its state in the energy basis* (akin to a measurement) — this is described by the single Lindblad operator $\hat{L}_2 = \sqrt{\gamma_2}\,\hat{a}^\dagger \hat{a}$. Here, $\gamma_2 = 1/T_2$ is the dephasing rate of the oscillator.

(a) Derive the equation of motion for the density matrix elements in the number basis[2], $\rho_{n,m} = \langle n|\hat{\rho}|m\rangle$, for a zero-temperature bath, and show that it is of the form:

$$\frac{d\rho_{n,m}}{dt} = -i\omega_0(n-m)\rho_{n,m} - \left(\frac{n+m}{2T_1} + \frac{(n-m)^2}{2T_2}\right)\rho_{n,m} + \frac{\sqrt{(n+1)(m+1)}}{T_1}\rho_{n+1,m+1}$$

(b) For a density matrix diagonal in the number basis $|n\rangle$, $\rho_{n,m} = 0$ if $n \neq m$ and $p_n = \rho_{n,n}$. Show that the resulting equation of motion is analogous to that of a classical birth-death process. What are the transition probabilities in and out of the state $|n\rangle$?

(c) Consider the case where the oscillator only undergoes energy relaxation ($\gamma_2 = 0$). Show that the canonical thermal state $\hat{\rho}_{th} \propto \exp\left(-\hat{H}_0/k_B T\right)$ is an equilibrium solution of the master equation. What is the population distribution in such an equilibrium?

(d) Show that for the Fock state $|n\rangle$, the decoherence time is T_1/n.

(e) For the energy and phase damped oscillator ($\gamma_1, \gamma_2 \neq 0$), derive the equation of motion for $\langle \hat{a} \rangle$ and $\langle \hat{a}^\dagger \hat{a} \rangle$.

11.2 Quantum master equation for a two-level system[3]

We consider a two-level system, with Hamiltonian $\hat{H}_0 = \frac{1}{2}\hbar\omega_0\sigma_z$. We assume only energy relaxation i.e. a master equation with the sole jump operator $\hat{L} = \sqrt{\gamma}\,\hat{\sigma}_-$, which corresponds to coupling with a zero-temperature bath, and want to express the equation of motion for the two-level system density operator.

[1] See Carmichael, "Statistical Methods in Quantum Optics 1", section 1.4.
[2] cf. Martinis, Phys. Rev. Lett. **103**, 200404 (2009) for an experimental investigation.
[3] Carmichael, "Statistical Methods in Quantum Optics 1", section 2.2.

(a) Derive the equations of motion for the operators, $\langle\hat{\sigma}_z\rangle$, $\langle\hat{\sigma}_+\rangle$, $\langle\hat{\sigma}_-\rangle$. These are known as the *optical Bloch equations*.

(b) How are the components of the density matrix related to the above expectation values?

(c) Show that the diagonal terms of the density matrix decay at double the rate of the off-diagonal elements for a zero-temperature bath and pure energy relaxation.

(d) Assume now that the two-level system is driven by an external (classical) radiation field. The coupling hamiltonian takes the form, $\hat{H}_{drive} = \frac{\hbar\Omega}{2}\left(\hat{\sigma}_+ e^{-i\Omega_L t} + h.c.\right)$. Write down the Bloch equations for this situation.

11.3 Numerical simulation of the decoherence of a nonclassical state of a harmonic oscillator

Using QuTiP package simulate the quantum mechanical decoherence of a nonclassical state of motion, i.e. simulate the decay of a harmonic oscillator quantum state in the presence of pure energy relaxation for the initial state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |3\rangle)$. This state is a superposition of a vacuum state and a state having energy $3\hbar\omega$. Assume a zero temperature bath, which allows to truncate the Hilbert space for $n > 3$. Simulate the evolution of the density matrix components ρ_{00}, ρ_{11}, etc. . . using the master equation derived in Ex1 above. Your result should model the results of the experiments carried out by Martinis[4].

11.4 Two-level system as a spectrum analyser (*)

Consider a two-level system whose free evolution is determined by the hamiltonian

$$\hat{H} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z.$$

It is coupled to an external "field" $f(t)$ via a dipole-like interaction,

$$\hat{H}_{int} = \hbar g\, f(t)\hat{\sigma}_x.$$

We will assume that $g \ll \omega_0$, so that the system can be analyzed perturbatively.

(a) Assume that the two-level system is prepared in its ground state $|0\rangle$ at time $t = 0$. Using the Schroedinger equation show that the probability amplitude for the system to be in its excited state at time t is given by,

$$\langle 1|\psi(t)\rangle = -ig\int_0^t f(t')e^{-i\omega_0 t'}\,dt' + \mathcal{O}(g^2).$$

(b) Thus, show that the average $\mathbb{E}$ (over realizations of the possibly random "field" f) probability to find the system in the excited state is given by,

$$p_1(t) = \mathbb{E}\left[|\langle 1|\psi(t)\rangle|^2\right] \approx g^2\int_0^t e^{i\omega_0(t'-t'')}\mathbb{E}\left[f(t')f(t'')\right]\,dt'\,dt''.$$

(c) Assume now that f is weak and stationary. Show that under this condition, the average excitation probability takes the form,

$$p_1(t) \approx g^2 t\cdot S_{ff}(\omega_0),$$

where $S_{ff}(\omega_0)$ is the double-sided spectral density of the field f. The transition rate to the upper state, $\Gamma_1 = \dot{p}_1(t) = g^2\, S_{ff}(\omega_0)$.

Thus by measuring the transition rate of a tunable two-level system, one can measure the quantum noise of externally coupled variables.

[4] See Figure 1C of "Decoherence Dynamics of Complex Photon States in Superconducting Circuits", Physical Review Letters, 103, 200404 (2009).