Statistical Physics IV: Non-equilibrium statistical physics

ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Exercise No.10

10.1 Standard quantum limit for gravitational wave detection [1]

(This exercise has 20 points)

In interferometric gravitational waves detectors (LIGO, Virgo, GEO600) the tiny displacements of test masses are inferred from the phase modulation of laser light, reflected from these masses. The equivalent scheme of gravitational waves detector is a one-sided optical cavity, one mirror of which is mounted on a spring that allows it to move as a harmonic oscillator with frequency Ω_m and decay rate Γ_m. Displacement x of the movable mirror modulates the optical cavity frequency ω_c as

$$\omega_c = (\omega_0 + \sqrt{2} g_0\, \hat{x}),$$

thus resulting in phase-modulation of the intracavity field $\hat{a}$. The field $\hat{a}$ leaks out of the cavity with the decay rate κ, creating the output optical field $\hat{a}_{out}$,

$$\hat{a}_{\text{out}} = \hat{a}_{\text{in}} - \sqrt{\kappa}\hat{a},$$

which can be directly detected.

The effect of gravitational waves in such picture is equivalent to a weak force $F_{GW}(t)$ applied to the test mass, which excites motion of the mass $\hat{x}(t)$ and can be detected as phase modulation of the output field $\hat{a}_{out}$.

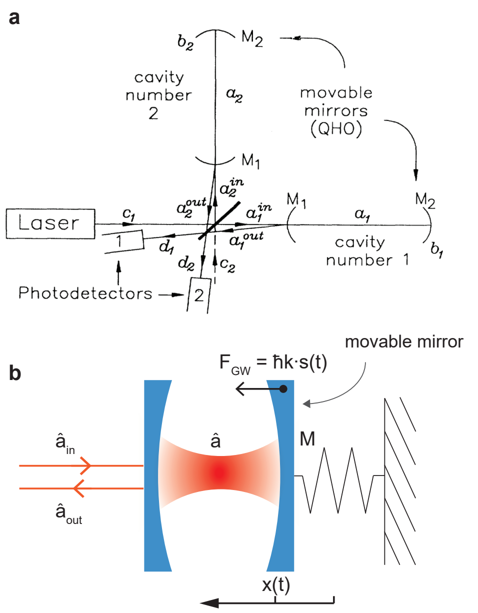

Figure 1: a) A realistic gravitational wave detector scheme b) The equivalent model of gravitational waves detector

Sensitivity of the real-life gravitational wave detectors are limited by many factors, the most important of which are the thermal noise, seismic noise at low frequencies and the laser shot noise at high frequencies. However, there is also a fundamental limitation on the precision of force measurements, the Standard Quantum Limit (SQL), which the present exercise derives. In state of the art gravitational waves detectors the contribution of this quantum noise can be at the level of percents.

1. Write down Hamiltonian for the system shown at the Fig. 10.1(b) including the energies of the optical field, mechanical oscillator, their interaction and the classical force $F_{GW}(t)$ ($\langle F_{GW}(t)\rangle = 0$) acting on the oscillator. It is convenient work in terms of the creation/annihilation operators for the optical field ($\hat{a}$) and dimensionless position ($\hat{x}$) and momentum ($\hat{p}$) of the harmonic oscillator,

$$\hat{x} = \frac{\hat{b} + \hat{b}^\dagger}{\sqrt{2}} \qquad \hat{p} = \frac{\hat{b} - \hat{b}^\dagger}{i\sqrt{2}},$$

where $\hat{b}$ is the annihilation operator for phonons. In such notations $F_{GW}(t) = \hbar k s(t)$.

2. Assume that a strong coherent laser drive, at the frequency ω_L, is applied to the system. In the reference frame, rotating with the laser frequency for the optical field, expand the

[1] See the seminal article "Quantum limits in interferometric detection of gravitational radiation" by Pace *et al.* and Walls, "Quantum Optics", chapter 8.3.

Hamiltonian around the average coherent values of operators

$$\hat{a} = \alpha + \delta\hat{a},$$
$$\hat{x} = \bar{x} + \delta\hat{x},$$
$$\hat{p} = \delta\hat{p},$$

where α, $\bar{x}$ are real numbers and $\delta\hat{a}$, $\delta\hat{x}$, $\delta\hat{p}$ contain the quantum flucutations. $\bar{x}$ is the average shift of the mirror position due to radiation pressure.

In a steady-state from the known α find $\bar{x}$ and comment on why the average momentum displacement $\bar{p} = 0$. Also show that the linearized interaction part of the Hamiltonian is

$$H_{\text{int}} = \sqrt{2}\hbar g(\delta\hat{a} + \delta\hat{a}^\dagger)\delta\hat{x},$$

where $g = g_0\alpha$ is the loaded cooperativity.

3. Show that the detection imprecision S_{xx}^{imp} caused by laser shot noise and the radiation pressure force exerted onto the mechanics S_{FF} ($F = -\frac{\partial H}{\partial x}$) obeys the relation $S_{FF}S_{xx}^{\text{imp}} \leq \hbar^2/4$. Further show that the measurement rate $\Gamma_{\text{meas}} = \frac{1}{2S_{xx}^{\text{imp}}}$ is equal to or less than the back-action rate $\Gamma_{\text{BA}} = \frac{2}{\hbar^2}S_{FF}$.

4. Assume that the laser drive frequency is set on resonance, i.e. such that the steady state detuning from the shifted cavity frequency is zero,

$$\omega_L = \omega_0 + \sqrt{2}g_0\bar{x}.$$

Write the quantum Langevin equations for the position ($\hat{x}$) and momentum ($\hat{p}$) of the mechanical oscillator and phase ($\hat{Y}$) and amplitude ($\hat{X}$) quadratures of the optical field, defined as[2]

$$\delta\hat{X} = \frac{\delta\hat{a} + \delta\hat{a}^\dagger}{\sqrt{2}} \qquad \delta\hat{Y} = \frac{\delta\hat{a} - \delta\hat{a}^\dagger}{i\sqrt{2}}$$

As derived in the lectures and earlier exercises include dissipation for the optical cavity coupled to the input field $\hat{a}_{\text{in}}$ through a partially reflecting mirror at a rate κ and the mechanics energy relaxation rate Γ_m. Neglect noise input for the mechanical oscillator.

5. The Langevin equations can be solved in Fourier domain. First express the intracavity fluctuations of the field $\delta\hat{X}[\omega]$ and $\delta\hat{Y}[\omega]$ in terms of the optical input noises $\delta\hat{X}_{\text{in}}[\omega]$ and $\delta\hat{Y}_{\text{in}}[\omega]$ and the mechanical motion $\delta\hat{x}[\omega]$. Show that only the phase quadrature of the field is modulated by the mechanical motion and thus carries information about the force, acting on the oscillator.

6. Express the motion of the oscillator $\delta\hat{x}[\omega]$ in terms of the optical input noises $\delta\hat{X}_{\text{in}}[\omega]$ and $\delta\hat{Y}_{\text{in}}[\omega]$ and the gravitational force $ks(t)$. Introduce the mechanical susceptibility to force $\chi_m[\omega] = \Omega_m/((i\omega - \Gamma_m/2)^2 + \Omega_m^2)$ for the brevity of notations, so that

$$\delta\hat{x}[\omega] = \chi_m[\omega](F_{GW}[\omega] + F_n[\omega]).$$

The noise force $F_n[\omega]$ due to the optical field (Quantum back-action of measurements or Radiation Pressure Shot Noise) is contaminating the measurements of $s(t)$.

7. In an experiment it is the output field $\delta\hat{Y}_{\text{out}}[\omega]$ that is being detected. Give the expression for such fluctuations

$$\delta\hat{Y}_{\text{out}}[\omega] = \delta\hat{Y}_{\text{in}}[\omega] - \sqrt{\kappa}\delta\hat{Y}[\omega] = A[\omega]\delta\hat{Y}_{\text{in}}[\omega] + B[\omega]\delta\hat{X}_{\text{in}}[\omega] + C[\omega]s[\omega]. \qquad (1)$$

[2]Note that different conventions exist for the definitions of the quadrature operators in the literature.

8. The spectrum of the detected signal is given by the symmetrized power spectrum of $\delta\hat{Y}_{\text{out}}$

$$S^{Y}_{\text{out}}[\omega] = \frac{1}{2}\left(\left\langle \left(\delta\hat{Y}_{\text{out}}[\omega]\right)^{\dagger}\delta\hat{Y}_{\text{out}}[\omega]\right\rangle + \left\langle \delta\hat{Y}_{\text{out}}[\omega]\left(\delta\hat{Y}_{\text{out}}[\omega]\right)^{\dagger}\right\rangle\right), \quad (2)$$

in order to compute which one needs to know the correlations of the incoming light. For a coherent laser drive

$$\left\langle \delta\hat{a}_{\text{in}}(t)\left[\delta\hat{a}_{\text{in}}(t')\right]^{\dagger}\right\rangle = \delta(t-t')$$
$$\left\langle \left[\delta\hat{a}_{\text{in}}(t)\right]^{\dagger}\delta\hat{a}_{\text{in}}(t')\right\rangle = 0.$$

Give the expressions for $\left\langle \left(\delta\hat{Y}_{\text{in}}[\omega]\right)^{\dagger}\delta\hat{Y}_{\text{in}}[\omega']\right\rangle$, $\left\langle \left(\delta\hat{X}_{\text{in}}[\omega]\right)^{\dagger}\delta\hat{X}_{\text{in}}[\omega']\right\rangle$, $\left\langle \left(\delta\hat{Y}_{\text{in}}[\omega]\right)^{\dagger}\delta\hat{X}_{\text{in}}[\omega']\right\rangle$ (note that e.g. $\left(\delta\hat{Y}[\omega]\right)^{\dagger} \neq \delta\hat{Y}^{\dagger}[\omega]$).

Argue why the cross correlators between $\delta\hat{Y}_{\text{in}}$ and $\delta\hat{X}_{\text{in}}$ do not contribute to the experimentally relevant symmetrized power spectrum. The latter means that the spectrum of the phase fluctuations will depend only on the squared amplitudes of the quantities $A(\omega)$, $B(\omega)$, $C(\omega)$ calculated above, and not on any cross products.

Calculate the spectrum in the eq. (2).

9. Returning to the expression for $\delta\hat{Y}_{\text{out}}[\omega]$ (Eq. 2), identify the terms which correspond to the gravitational signal, laser shot noise in the detection and the measurement back action noise, caused by the fluctuation of the laser field driving the oscillator.

10. From the output signal given by the Eq. 1 the gravitational waves signal ($s(t)$) can be estimated as

$$s_{\text{est}}[\omega] = s[\omega] + \frac{A[\omega]}{C[\omega]}\delta\hat{Y}_{\text{in}}[\omega] + \frac{B[\omega]}{C[\omega]}\delta\hat{X}_{\text{in}}[\omega]. \quad (3)$$

The detection noise is given by the last two terms, which have different scaling with input optical power ($P_{\text{in}} \propto \alpha^2 \propto g^2$). Show that there exists an optimum input power for which the noise spectral density is minimum and find this power (in terms of g). Adopt here the approximation that all the Fourier frequencies of interest is within the cavity linewidth ($\omega \ll \kappa$), which is the most favorable situation for interferometric measurements. Show that the minimum noise spectral density is given by

$$S_{\text{noise,SQL}}[\omega] = \frac{|A[\omega]|^2}{|C[\omega]|^2}S^{Y}_{\text{in}} + \frac{|B[\omega]|^2}{|C[\omega]|^2}S^{X}_{\text{in}} = \frac{1}{k^2|\chi_m[\omega]|}. \quad (4)$$

This is the standard quantum limit (SQL) for a linear measurement. If the laser power is too low, the output is dominated by the shot noise, and if it is too high, the quantum intensity fluctuations drive the mechanical oscillator and drown the signal to be measured.

10.2 Quantum regression theorem and photon bunching[3]

For a set of operators $\hat{A}_{\mu}(t)$ evolving under the equations of motion

$$\frac{d}{dt}\left\langle \hat{A}_{\mu}(t)\right\rangle = \sum_{\nu} M_{\mu\nu}\left\langle \hat{A}_{\nu}(t)\right\rangle$$

the quantum regression theorem takes place:

$$\frac{d}{d\tau}\left\langle \hat{O}_1(t)\hat{A}_{\mu}(t+\tau)\hat{O}_2(t)\right\rangle = \sum_{\nu} M_{\mu\nu}\left\langle \hat{O}_1(t)\hat{A}_{\nu}(t+\tau)\hat{O}_2(t)\right\rangle,$$

[3] Carmichael, "Statistical Methods in Quantum Optics 1", section 1.5.

where $\hat{O}_{1,2}(t)$ are arbitrary system operators.

Consider an energy-damped harmonic oscillator with dissipation constant γ. Under these conditions using the regression theorem show that the photon statistics is an example of photon bunching[4] such that

$$\left\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\right\rangle = \bar{n}^2\left(1+e^{-\gamma t}\right).$$

10.3 Asymmetry of the spectral density of the quantum harmonic oscillator [5]

Consider a simple quantum harmonic oscillator with mass m and frequency Ω. The oscillator is at a temperature T; this temperature is maintained through an infinitesimal coupling to a heat bath (therefore, one can neglect the energy decay rate κ of the oscillator). Let $\hat{x}$ and $\hat{p}$ denote the position and momentum operators (obeying the canonical commutation relation) and $\hat{a}$, $\hat{a}^\dagger$ the standard annihilation and creation operators.

1. Show that the autocorrelation function of the position operator is given by

$$C_{xx}(t) \equiv \langle \hat{x}(t)\hat{x}(0)\rangle = \langle \hat{x}(0)\hat{x}(0)\rangle \cos(\Omega t) + \langle \hat{p}(0)\hat{x}(0)\rangle \frac{1}{M\Omega}\sin(\Omega t)$$

2. Show that in thermal equilibrium the following expressions hold: $\langle \hat{x}(0)\hat{p}(0)\rangle = i\hbar/2$ and $\langle \hat{p}(0)\hat{x}(0)\rangle = -i\hbar/2$

3. Using these expressions, show that the autocorrelation function is given by

$$C_{xx}(t) \propto \bar{n}(\hbar\Omega)e^{i\Omega t} + [\bar{n}(\hbar\Omega)+1]e^{-i\Omega t},$$

   where $\bar{n}$ is the Bose-Einstein occupation factor. Calculate the proportionality factor.

4. Using this correlation function, calculate the spectral density $S_{xx}(\omega)$ and show that it is asymmetric in frequency. Show that in the high temperature limit ($k_B T >> \hbar\Omega$), this spectral density becomes symmetric (thus, it coincides with the classical case).

We note that the positive-frequency part of the spectral density is a measure of the ability of the oscillator to absorb energy, while the negative-frequency part is a measure of the ability of the oscillator to emit energy.

10.4 Asymmetry of the spectral density of the quantum harmonic oscillator with damping (*)[6]

We will repeat the calculation of the spectral density of a quantum harmonic oscillator, but now taking damping into consideration. Therefore, we start from the quantum Langevin-equation

$$\frac{d}{dt}\hat{a} = -i\Omega\hat{a} - \frac{\kappa}{2}\hat{a} - \sqrt{\kappa}\hat{F}_{\text{in}}(t)$$

Here, $\hat{F}_{\text{in}}$ is the input noise operator which satisfies the following time-domain correlation relations:

$$\langle \hat{F}_{\text{in}}^\dagger(t)\hat{F}_{\text{in}}(t')\rangle = \bar{n}\delta(t-t')$$
$$\langle \hat{F}_{\text{in}}(t)\hat{F}_{\text{in}}^\dagger(t')\rangle = (\bar{n}+1)\delta(t-t')$$

[4] Two photons are more likely to be detected at the same time. This effect was used by Hanbury Brown and Twiss to measure the apparent solid angle of stars, see "A test of a new type of stellar Interferometer on Sirius", Nature 1956.

[5] see Clerk et al: Introduction to quantum noise, measurement and amplification, RMP 82 (2010) Sec. 2 on quantum noise spectra

[6] This is a bonus exercise

1. Write down the correlation relations for the input noise operators written above in the frequency domain.
2. By solving the Langevin-equation for $\hat{a}$ and $\hat{a}^\dagger$ in the frequency domain, calculate the spectral densities $S_{a^\dagger a}(\omega) = \int_{-\infty}^{\infty} \langle \hat{a}^\dagger(t+\tau)\hat{a}(t)\rangle e^{i\omega\tau} d\tau$ and $S_{aa^\dagger}(\omega)$ (defined analogously).
3. Next, express $S_{xx}(\omega)$ in terms of $S_{a^\dagger a}(\omega)$ and $S_{aa^\dagger}(\omega)$ and show that indeed, the position spectral density of a damped quantum harmonic oscillator is asymmetric in frequency.