Statistical Physics IV: Non-equilibrium statistical physics
ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Solutions to Exercise No.9

Solution: Quantization of an electrical LC circuit

(a) $H = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}\frac{\Phi^2}{L}$

(b) By ensuring $[a, a^\dagger] = 1$, one gets a first equation for the factors of interest. To obtain the second equation, one has multiple options. First, one can make sure that a gets a meaningful dimension. Second, one can compute the Hamiltonian and cancel the terms aa, $a^\dagger a^\dagger$. Then one gets

$$\hat{\Phi} = \Phi_{zpf}(\hat{a} + \hat{a}^\dagger) \qquad \Phi_{zpf} = \sqrt{\frac{\hbar}{2}\sqrt{\frac{L}{C}}}$$

$$\hat{Q} = Q_{zpf}\frac{(\hat{a} - \hat{a}^\dagger)}{i} \qquad Q_{zpf} = \sqrt{\frac{\hbar}{2}\sqrt{\frac{C}{L}}}$$

$$\hat{a} = \frac{1}{2}\left(\frac{\hat{\Phi}}{\Phi_{zpf}} + i\frac{\hat{Q}}{Q_{zpf}}\right)$$

One should then derive

$$H = \hbar\frac{1}{\sqrt{LC}}(a^\dagger a + \frac{1}{2})$$

(c) One obtains the zero point fluctuations derived above.

Solution: The Sudarshan-Glauber representation of a density matrix

1.

$$\int d^2\alpha\, |\alpha\rangle\langle\alpha| = \sum_{n,m}\frac{1}{\sqrt{n!m!}}|n\rangle\langle m| \underbrace{\int d^2\alpha\, e^{-|\alpha|^2}\alpha^n(\alpha^*)^m}_{(\star)} \tag{1}$$

$$(\star) = \int_0^\infty dr\, e^{-r^2} r^{n+m} \int_0^{2\pi} d\phi\, e^{i\phi(n-m)} \tag{2}$$

$$\overset{r^2=s}{=} 2\pi\delta(n-m)\cdot\frac{1}{2}\int_0^\infty ds\, e^{-s}s^{\frac{n+m}{2}} \tag{3}$$

$$= \pi\delta(n-m)\left(\frac{n+m}{2}\right)! \tag{4}$$

$$\Rightarrow \int\frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| = \frac{1}{\pi}\pi\sum_n|n\rangle\langle n| = \mathbb{1} \tag{5}$$

2.

$$\langle\alpha|\,|\beta\rangle = e^{-\frac{|\alpha|^2}{2}}e^{-\frac{|\beta|^2}{2}}\sum_{n,m}\frac{(\alpha^*)^n\beta^m}{\sqrt{n!m!}}\langle n|\,|m\rangle = e^{-\frac{|\alpha|^2}{2}}e^{-\frac{|\beta|^2}{2}}e^{\alpha^*\beta} \tag{6}$$

$$\Rightarrow |\alpha\rangle = \int\frac{d^2\beta}{\pi}|\beta\rangle\langle\beta|\,|\alpha\rangle = \frac{e^{-|\alpha|^2/2}}{\pi}\int e^{-|\beta|^2/2+\beta^*\alpha}\,|\beta\rangle\; d^2\beta \tag{7}$$

3. This is trivial

$$\hat{O} = \mathbb{1}\hat{O}\mathbb{1} = \int O(\alpha,\beta)\,|\alpha\rangle\langle\beta|\; d^2\alpha\, d^2\beta, \tag{8}$$

4.

$$O(\alpha,\beta) = \frac{1}{\pi^2}\int d^2\alpha' \int d^2\beta' \langle\alpha|\,|\alpha'\rangle \frac{\langle\alpha'|\,\hat{O}\,|\beta'\rangle}{\pi^2} \langle\beta'|\,|\beta\rangle \tag{9}$$

$$= \frac{e^{-(|\alpha|^2+|\beta|^2)/2}}{\pi^2}\int O(\alpha',\beta')e^{-(|\alpha'|^2+|\beta'|^2)/2+\alpha^*\alpha'+\beta(\beta')^*} d^2\alpha'\; d^2\beta' \tag{10}$$

5.

$$\hat{O} = \int P(\beta)\,|\beta\rangle\,\langle\beta|\,d^2\beta \tag{11}$$

$$\langle-\alpha|\,\hat{O}\,|\alpha\rangle = \int d^2\beta\, P(\beta)e^{-(|\alpha|^2+|\beta|^2)/2}e^{\beta^*\alpha-\alpha^*\beta}e^{-(|\alpha|^2+|\beta|^2)/2} \tag{12}$$

So,

$$\int d^2\alpha\, \langle-\alpha|\,\hat{O}\,|\alpha\rangle\, e^{|\alpha|^2+|\beta|^2+\alpha^*\beta'-\beta'^*\alpha} \tag{13}$$

$$= \int d^2\beta \int d^2\alpha\, P(\beta)e^{|\beta'|^2-|\beta|^2}e^{\alpha^*(\beta'-\beta)-\alpha(\beta'^*-\beta^*)} \tag{14}$$

$$= \int d^2\beta\, P(\beta)e^{|\beta'|^2-|\beta|^2}\int d^2\alpha\, e^{2i\,\mathrm{Im}[\alpha^*(\beta'-\beta)]} \tag{15}$$

$$= \int d^2\beta\, P(\beta)e^{|\beta'|^2-|\beta|^2}\int d^2\alpha\, e^{2i\,\mathrm{Re}[\alpha]\,\mathrm{Im}[\beta'-\beta]-2i\,\mathrm{Im}[\alpha]\,\mathrm{Re}[\beta'-\beta]} \tag{16}$$

$$= \int d^2\beta\, P(\beta)e^{|\beta'|^2-|\beta|^2}(2\pi)^2\delta(\mathrm{Im}[\beta']-\mathrm{Im}[\beta])\delta(\mathrm{Re}[\beta']-\mathrm{Re}[\beta]) \tag{17}$$

$$= \pi^2 P(\beta')\,. \tag{18}$$

$$\Rightarrow P(\beta) = \int O(-\alpha,\alpha)e^{|\alpha|^2+|\beta|^2+\alpha^*\beta-\beta^*\alpha}\; d^2\alpha. \tag{19}$$

Solution: Quantum Langevin equation for a harmonic oscillator interacting with a heat bath

1. The full interaction Hamiltonian will contain terms like:

$$\sum_k(\hat{a}+\hat{a}^\dagger)(\hat{b}_k-\hat{b}_k^\dagger) = \sum_k \hat{a}\hat{b}_k - \hat{a}\hat{b}_k^\dagger + \hat{a}^\dagger\hat{b}_k - \hat{a}^\dagger\hat{b}_k^\dagger\,. \tag{20}$$

   We can change reference frame (or, equivalently, go to the interaction picture) and do the following substitutions: $\hat{a} \to \hat{a}e^{-i\omega_s t}$ and $\hat{b} \to \hat{b}e^{-i\omega_k t}$. So that the terms above become:

$$\sum_k \hat{a}\hat{b}_k e^{-i(\omega_k+\omega_s)t} - \hat{a}\hat{b}_k^\dagger e^{i(\omega_k-\omega_s)t} + \hat{a}^\dagger\hat{b}_k e^{-i(\omega_k-\omega_s)t} - \hat{a}^\dagger\hat{b}_k^\dagger e^{i(\omega_k+\omega_s)t}\,. \tag{21}$$

   Now if $|\omega_k+\omega_s| \gg |\omega_k-\omega_s|$, the terms with $e^{\pm i(\omega_k+\omega_s)t}$ will average to zero over much shorter time scales than the terms with $e^{\pm i(\omega_k-\omega_s)t}$, thus the former terms can be neglected. This is the rotating wave approximation.

2. We calculate the equations of motion using $\partial_t\hat{O} = \frac{i}{\hbar}[H,\hat{O}]$. A straightforward calculation leads to:

$$\partial_t\hat{a} = -i\omega_s\hat{a} - i\sum_k g_k\hat{b}_k \tag{22}$$

$$\partial_t\hat{b}_k = -i\omega_k\hat{b}_k - ig_k\hat{a} \tag{23}$$

3. First we formally integrate the equation for $\hat{b}_k$. We obtain (can be verified by substitution or derived using variation of the constant):

$$\hat{b}_k(t) = \hat{b}_k(t)e^{-i\omega_k t} - ig_k \int_0^t \mathrm{d}t'\, \hat{a}(t')e^{i\omega_k(t-t')} \,. \tag{24}$$

We now insert this back into the expression for $\hat{a}$:

$$\partial_t \hat{a} = -i\omega_s \hat{a}(t) \underbrace{-i\sum_k g_k \hat{b}_k(0)e^{-i\omega_k t}}_{\hat{f}_a(t)} - \sum_k g_k^2 \int_0^t \mathrm{d}t'\, \hat{a}(t')e^{i\omega_k(t-t')} \,. \tag{25}$$

Now we transform the sum into an integral:

$$\sum_k g_k^2 \to \int_0^\infty \mathrm{d}\omega_k\, D(\omega_k)|g(\omega_k)|^2 \,; \tag{26}$$

and apply the 1st Markov approximation $g_k = g(\omega_k) = g$ and make the assumption that the density of states is slowly varying $D(\omega) = D(\omega_s)$:

$$\sum_k g_k^2 \int_0^t \mathrm{d}t'\, \hat{a}(t')e^{-i\omega_k(t-t')} = \int_0^\infty \mathrm{d}\omega_k\, D(\omega_k)|g(\omega_k)|^2 \int_0^t \mathrm{d}t'\, \hat{a}(t')e^{-i\omega_k(t-t')} \tag{27}$$

$$= D(\omega)|g(\omega)|^2 \int_0^t \mathrm{d}t'\, \hat{a}(t') \int_0^\infty \mathrm{d}\omega_k\, e^{-i\omega_k(t-t')} \tag{28}$$

$$= D(\omega)|g(\omega)|^2 \int_0^t \mathrm{d}t'\, \hat{a}(t') 2\pi\delta(t-t') \tag{29}$$

$$= 2\pi D(\omega)|g(\omega)|^2 \frac{1}{2}\hat{a}(t) \equiv \frac{\kappa}{2}\hat{a}(t) \,. \tag{30}$$

Finally we obtain the QLE for $\hat{a}$:

$$\partial_t \hat{a}(t) = -i\omega_s \hat{a}(t) - \frac{\kappa}{2}\hat{a}(t) + \hat{f}_a(t) \,. \tag{31}$$

4. Take an operator $\hat{a}$ whose equation of motion is given by $\partial_t \hat{a} = \frac{i}{\hbar}[H,\hat{a}]$. Then $\hat{\tilde{a}} = \hat{a}e^{-i\omega t}$ has the equation of motion:

$$\partial_t \hat{\tilde{a}} = i\omega\hat{\tilde{a}} + \frac{i}{\hbar}\left[H,\hat{\tilde{a}}\right] \,. \tag{32}$$

So with going to a frame rotating with ω_s ($\hat{a} = \hat{\tilde{a}}e^{i\omega_s t}$):

$$\partial_t \hat{\tilde{a}} = -\frac{\kappa}{2}\hat{\tilde{a}}(t) + \hat{f}_{\tilde{a}}(t) \tag{33}$$

5. $F(t) = \hat{f}_{\tilde{a}}(t)$

$$\left\langle F^\dagger(t)F(t')\right\rangle = \sum_k\sum_{k'} g_k g_{k'} e^{i\omega_k t - i\omega_{k'}t'} \left\langle \hat{b}_k^\dagger(0)\hat{b}_{k'}(0)\right\rangle \tag{34}$$

$$= 2\pi\sum_k g_k^2 \bar{n}_k e^{i\omega_k(t-t')} \tag{35}$$

$$= \int_0^\infty \mathrm{d}\omega\, g^2 \bar{n}(\omega) e^{i\omega_k(t-t')} D(\omega) \tag{36}$$

$$= \kappa\bar{n}_{\mathrm{th}}\delta(t-t') \tag{37}$$

6.

$$\partial_t\left[\hat{a}(t),\hat{a}^\dagger(t)\right] = -\kappa\left[\hat{a}(t),\hat{a}^\dagger(t)\right] + \left[F(t),\hat{a}^\dagger(t)\right] + \left[\hat{a}(t),F^\dagger(t)\right] \tag{38}$$

To compute $\left[\hat{a}(t),F^\dagger(t)\right]$ we use $\hat{a}(t) = \hat{a}(0)e^{-\frac{\kappa}{2}t} + \int\limits_0^t \mathrm{d}t'\, e^{-\frac{\kappa}{2}(t-t')}F(t')$ and obtain:

$$\left[\hat{a}(t),F^\dagger(t)\right] = \int\limits_0^t \mathrm{d}t'\, e^{-\frac{\kappa}{2}(t-t')}\left[F(t'),F^\dagger(t)\right] \tag{39}$$

$$= \frac{\kappa}{2} = \left[F(t),\hat{a}^\dagger(t)\right]. \tag{40}$$

So

$$\partial_t\left[\hat{a}(t),\hat{a}^\dagger(t)\right] = -\kappa\left[\hat{a}(t),\hat{a}^\dagger(t)\right] + \kappa = 0\ \forall t \tag{41}$$

knowing that $\left[\hat{a}(0),\hat{a}^\dagger(0)\right]$.

7.

$$\left\langle \partial_t\hat{a}\right\rangle = \partial_t\left\langle \hat{a}\right\rangle = -\frac{\kappa}{2}\left\langle \hat{a}(t)\right\rangle + \left\langle F(t)\right\rangle = -\frac{\kappa}{2}\left\langle \hat{a}(t)\right\rangle \Rightarrow \left\langle \hat{a}(t)\right\rangle = \left\langle \hat{a}(0)\right\rangle e^{-\frac{\kappa}{2}t} \tag{42}$$

8. Using all the information up until now it is a straightforward calculation to find the equation of motion

$$\langle t|N|t\rangle = -\kappa\left\langle N(t)\right\rangle + \left\langle F^\dagger(t)\hat{a}(t)\right\rangle + \left\langle \hat{a}^\dagger(t)F(t)\right\rangle = -\kappa\left\langle N(t)\right\rangle + \kappa\bar{n}_{\mathrm{th}} \tag{43}$$

which solves to:

$$\left\langle N(t)\right\rangle = \left(\left\langle N(0)\right\rangle - \bar{n}_{\mathrm{th}}\right)e^{-\kappa t} + \bar{n}_{\mathrm{th}} \tag{44}$$