Quantum Field Theory

Training Exercises

Exercise 1

Consider the following lagrangian for three real scalars:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2}M^2\Phi^2 + \frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1 + \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{1}{2}\lambda_1 \Phi \phi_1^2 - \frac{1}{2}\lambda_2 \Phi \phi_2^2 \,.$$

- What is the dimensionality of the couplings λ_1, λ_2 ?
- Compute the amplitude of the processes $\phi_1\phi_1 \to \phi_1\phi_1$, $\phi_2\phi_2 \to \phi_2\phi_2$, $\phi_1\phi_2 \to \phi_1\phi_2$.
- Under which conditions are the first two amplitudes equal ? What can we say, in that case, at the level of the lagrangian ?

Now consider the following lagrangian for three real scalars Φ_1, Φ_2, Φ_3, and 2 complex scalars ϕ_1, ϕ_2:

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\sum_{A=1}^{3} \partial_\mu \Phi_A \partial^\mu \Phi_A + \frac{1}{2}M^2 \sum_{A=1}^{3} \Phi_A^2 + \partial_\mu \phi_1^\dagger \partial^\mu \phi_1 + \partial_\mu \phi_2^\dagger \partial^\mu \phi_2 \\ &- \lambda_1 \Phi_1 (\phi_2^\dagger \phi_1 + \phi_1^\dagger \phi_2) - i\lambda_2 \Phi_2 (\phi_2^\dagger \phi_1 - \phi_1^\dagger \phi_2) - \lambda_3 \Phi_3 (|\phi_1|^2 - |\phi_2|^2)\,.\end{aligned}$$

- Compute the total decay rates of the three massive scalars Φ_1, Φ_2, Φ_3.
- Check that they are equal if $\lambda_1 = \lambda_2 = \lambda_3$.
- Can you interpret this result ?
- Will the decay rates still be equal at higher order ?

Exercise 2

Consider a quantum field theory with one dirac fermion ψ and two real scalars ϕ_1, ϕ_2, and lagrangian

$$\mathcal{L} = \bar{\psi}(i\,\partial\!\!\!/ - m)\psi + \frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1 + \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2 - g\phi_1 \partial_\mu \phi_2 \bar{\psi}\gamma^\mu \psi \,.$$

- What is the dimensionality of the coupling g ?
- Compute the following amplitudes

$$\mathcal{M}(\phi_1 \psi \to \phi_2 \psi), \quad \mathcal{M}(\phi_1 \phi_2 \to \bar{\psi}\psi), \quad \mathcal{M}(\bar{\psi}\psi \to \phi_1\phi_2)\,.$$

- Compute the total unpolarized cross-section for the first process $\sigma(\phi_1\psi \to \phi_2\psi)$.
- What changes if $\phi_1 = \phi_2$?

Exercise 3

Consider a theory with a massive complex scalar field ϕ with charge e and a massless real vector field A_μ (photon).
(a) Write the most general Lagrangian for this theory up to dimension $d \leq 4$ and specify the dimension of the parameters of your theory.
(b) Compute the Feynman rules for this theory.
(c) Draw all the Feynman diagrams that contribute to the scattering $\phi\gamma \to \phi\gamma$.
(d) Compute the amplitude for this process and show that the Ward identity is satisfied.
(e) Compute the unpolarized differential cross section for this process in the rest frame of the initial ϕ. Compute then the total cross section in the limit where the energy of the incoming photon is smaller then the mass of the scalar.

Exercise 4

Consider the decay of a massive Dirac fermion Ψ (of mass M) through the interaction lagrangian

$$\mathcal{L} = \lambda \bar{\psi}_L \Psi_R \phi + \text{h.c.}$$

where ψ and ϕ a massless Dirac field and a massless real scalar respectively and λ is the coupling constant.
(a) What is the dimensionality of λ?
(b) Compute $|\mathcal{M}|^2$, summing (averaging) over polarizations of the final (initial) state particles.
(c) Compute the total decay width Γ of the field Ψ.
If you cannot finish the computation, at least estimate the dependence of Γ upon M and λ by dimensional analysis.