Quantum Field Theory

Solutions Set 6

Exercise 1

Consider a vector field A^μ at the origin $x^\mu = 0$. Under a Lorentz transformation this transforms as

$$U(\Lambda)A^\mu(0)U^\dagger(\Lambda) = \Lambda^\mu{}_\nu A^\nu(0)\,,$$

where Λ is the usual 4×4 representation of the Lorentz group and $U(\Lambda)$ is the representation of the same transformation in the Hilbert space of the particle states. We can expand both sides to the first order in the transformation parameters. Remember the exponential forms

$$\begin{aligned} U(\Lambda) &= \exp(-i\boldsymbol{\theta}\cdot\mathbf{J} - i\boldsymbol{\eta}\cdot\mathbf{K}) \\ \Lambda &= \exp(-i\boldsymbol{\theta}\cdot\mathcal{J} - i\boldsymbol{\eta}\cdot\mathcal{K})\,, \end{aligned}$$

where $\mathbf{J}$ and $\mathbf{K}$ are operators in the Hilbert space while $\mathcal{J}$ and $\mathcal{K}$ are 4×4 matrices given by

$$(\mathcal{M}^{\alpha\beta})_{\mu\nu} = i(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu)$$

and

$$\begin{aligned} \mathcal{J}^i &= \frac{1}{2}\epsilon^{ijk}\mathcal{M}^{jk} \\ \mathcal{K}^i &= \mathcal{M}^{0i}. \end{aligned}$$

Expanding both sides of the Lorentz transformation equation we find that for rotations

$$[J^i, A^\mu(0)] = \frac{i}{2}\epsilon^{ijk}(\eta^{j\mu}A^k(0) - \eta^{k\mu}A^j(0))$$

and for boosts

$$[K^i, A^\mu(0)] = i(\eta^{0\mu}A^i(0) - \eta^{i\mu}A^0(0))\,.$$

We can separate the time and spatial component of $A^\mu(0)$ to get easier formulas:

$$\begin{aligned} [J^i, A^0(0)] &= 0 \\ [J^i, A^j(0)] &= -i\epsilon^{ijk}A^k(0) \\ [K^i, A^0(0)] &= iA^i(0) \\ [K^i, A^j(0)] &= i\delta^{ij}A^0(0) \end{aligned} \tag{1}$$

These results are what we would expect: a rotation doesn't affect the time component of the vector and it mixes the spacial components with an ϵ tensor, while a boost mixes the time component with one of the spacial components.

Now let's do a parity transformation. On the Hilbert space this is implemented through an unitary operator U_P. Since parity acts linearly, this will be implemented by a 4×4 matrix in the index space as follows

$$U_P A^\mu(0) U_P = \mathscr{P}^\mu{}_\nu A^\nu(0)\,.$$

So far $\mathscr{P}$ is an arbitrary 4×4 matrix, but this equality and the commutators previously computed will fix its components up to an overall factor. For example take the first commutator and multiply by U_P left and right

$$\begin{aligned} 0 = U_P[J^i, A^0(0)]U_P &= [U_P J^i U_P, U_P A^0(0) U_P] \\ &= [J^i, U_P A^0(0) U_P] = \mathscr{P}^0{}_\mu[J^i, A^\mu(0)] \\ &= \mathscr{P}^0{}_0[J^i, A^\mu(0)] + \mathscr{P}^0{}_j[J^i, A^j(0)] = -i\epsilon^{ijk}\mathscr{P}^0{}_j A^k(0) \end{aligned}$$

from which we deduce that $\mathscr{P}^0{}_j = 0$. Similarly from the second commutator

$$[J^i, U_P A^j(0) U_P] = -i\epsilon^{ijk} U_P A^k(0) U_P \,.$$

The left hand side is

$$\mathscr{P}^j{}_\mu [J^i, A^\mu(0)] = \mathscr{P}^j{}_0 [J^i, A^0(0)] + \mathscr{P}^j{}_k [J^i, A^k(0)] = -i\epsilon^{ikl} \mathscr{P}^j{}_k A^l(0)$$

while the right hand side

$$-i\epsilon^{ijk} \mathscr{P}^k_\mu A^\mu(0) = -i\epsilon^{ijk} \mathscr{P}^k_0 A^0(0) - i\epsilon^{ijk} \mathscr{P}^k_l A^l(0) \,.$$

Since the left hand side is independent of A^0 we must have $\mathscr{P}^k_0 = 0$. Finally we use the third commutator

$$\mathscr{P}^0_0 [K^i, A^0] = i\mathscr{P}^0_0 A^i = -i\mathscr{P}^i_j A^j$$

From which we see that $\mathscr{P}^i_j \propto \delta^i_j$ and $\mathscr{P}^i_i = -\mathscr{P}^0_0$. So we find that

$$\mathscr{P} = \mathrm{diag}(a, -a, -a, -a).$$

Another way of deriving part of this result would be to exploit the fact that parity commutes with rotations. In fact we have that applying a rotation on the definition of the $\mathscr{P}$ matrix

$$\begin{aligned} \mathscr{P}^\mu{}_\nu R^\nu{}_\rho A^\rho(0) &= \mathscr{P}^\mu{}_\nu U(R) A^\nu(0) U(R)^\dagger = U(R) U_P A^\mu(0) U_P U(R)^\dagger \\ &= U_P U(R) A^\mu(0) U(R)^\dagger U_P = U_P R^\mu{}_\nu A^\nu U_P \\ &= R^\mu{}_\nu \mathscr{P}^\nu{}_\rho A^\rho \,. \end{aligned}$$

So we see that the commutation between parity and rotations is also valid for the 4×4 matrices. This immediately implies that the most general form for $\mathscr{P}$ is

$$\mathrm{diag}(a, b, b, b) \,. \tag{2}$$

To see this remember that a 4×4 matrix (or if you want a two index Lorentz tensor) can be decomposed as two scalars (the $(0,0)$ component and the trace), three vectors (the $(0,i)$ and $(i,0)$ components and the antisymmetric part of the 3×3 lower-right block) and a rank-2 symmetric tensor (the traceless symmetric part of the 3×3 lower-right block). Since this matrix is invariant under rotation the only components that can be non-zero are the first component and the diagonal of the 3×3. Now it's only a matter to fix the relation between these two numbers through some other commutators as we did in the other case.

Requiring that the action of parity twice is the identity requires that a is ± 1.

To generalize this to the case where the field is evaluated at a point x^μ we use the translation operator

$$e^{iP\cdot x} A^\mu(0) e^{-iP\cdot x} = A^\mu(x)$$

where P^μ are the translation generators (not to be confused with the P for parity). Using this we can write

$$\begin{aligned} U_P A^\mu(x) U_P &= U_P e^{iP\cdot x} A^\mu(0) e^{-iP\cdot x} U_P \\ &= e^{iP\cdot x_P} U_P A^\mu(0) U_P e^{-iP\cdot x_P} \\ &= e^{iP\cdot x_P} \mathscr{P}^\mu{}_\nu A^\nu(0) e^{-iP\cdot x_P} \\ &= \mathscr{P}^\mu{}_\nu A^\nu(x_P) \end{aligned}$$

where we have used the property

$$U_P e^{iP\cdot x} U_P = e^{i(U_P P U_P)\cdot x} = e^{i(U_P P^0 U_P x_0 + U_P P^i U_P x_i)} = e^{i(P^0 x_0 - P^i x_i)} = e^{iP\cdot x_P} \,.$$

Now we do the same with time reversal. Again it will act linearly on the fields

$$U_T A^\mu(0) U_T = \mathscr{T}^\mu{}_\nu A^\nu(0)$$

where $\mathscr{T}$ is a 4×4 matrix. By the same argument as exposed above, we know that time reversal commutes with rotations, so the 4×4 matrices representing rotations and $\mathscr{T}$ must commute as well. As a consequence, we know that in this case too $\mathscr{T}$ will have the form (2).

Similarly as before, we apply U_T left and right of the commutators (1), remembering that U_T is antiunitary. The first one yields

$$\begin{aligned}0 &= U_T[J^i, A^0(0)]U_T = [U_T J^i U_T, U_T A^0(0) U_T] \\ &= [-J^i, U_T A^0(0) U_T] = -\mathscr{T}^0{}_\mu [J^i, A^\mu(0)] \\ &= -\mathscr{T}^0{}_0[J^i, A^\mu(0)] - \mathscr{T}^0{}_j[J^i, A^j(0)] = i\epsilon^{ijk}\mathscr{T}^0{}_j A^k(0)\end{aligned}$$

from which we conclude $\mathscr{T}^0{}_j = 0$. Similarly, from the third

$$\begin{aligned}U_T[K^i, A^0(0)]U_T &= [K^i, \mathscr{T}^0{}_\mu A^\mu(0)] = \mathscr{T}^0{}_\mu[K^i, A^\mu(0)] = \mathscr{T}^0{}_0[K^i, A^0(0)] = i\mathscr{T}^0{}_0 A^i(0) \\ = U_T(iA^i(0))U_T &= -iU_T A^i(0) U_T = -i\mathscr{T}^i{}_\mu A^\mu(0) = -i\mathscr{T}^i{}_0 A^0(0) - i\mathscr{T}^i{}_j A^j(0)\end{aligned}$$

where we have used $\mathscr{T}^0{}_i = 0$ in the third equality, (1) in the fifth, and antiunitarity in the sixth. Comparing the end of the two lines yields $\mathscr{T}^i{}_0 = 0$ and $\mathscr{T}^i{}_j = -\delta^{ij}\mathscr{T}^0{}_0$ as before. We get

$$\mathscr{T} = \mathrm{diag}(a, -a, -a, -a)$$

and the fact that time reversal square is zero yields $a = \pm 1$.

As above, we deduce what happens at point x^μ in a similar way

$$\begin{aligned}U_T A^\mu(x) U_T &= U_T e^{iP\cdot x} A^\mu(0) e^{-iP\cdot x} U_T \\ &= e^{-iP\cdot x_P} U_T A^\mu(0) U_T e^{iP\cdot x_P} \\ &= e^{-iP\cdot x_P} \mathscr{T}^\mu{}_\nu A^\nu(0) e^{iP\cdot x_P} \\ &= \mathscr{T}^\mu{}_\nu A^\nu(-x_P)\end{aligned}$$

where we used the following (remember U_T is antiunitary)

$$U_T e^{iP\cdot x} U_T = e^{-i(U_T P U_T)\cdot x} = e^{-i(U_T P^0 U_T x_0 + U_T P^i U_T x_i)} = e^{-i(P^0 x_0 - P^i x_i)} = e^{-iP\cdot x_P}\,.$$

Exercise 2

The time reversal transformation property for the scalar field is

$$T\phi(\vec{x}, t)T = \eta_T \phi(\vec{x}, -t),$$

which leads to

$$\begin{aligned}Ta(\vec{k})T &= \eta_T a(-\vec{k}), \\ Tb^\dagger(\vec{k})T &= \eta_T b^\dagger(-\vec{k}),\end{aligned}$$

for the annihilation and creation operators.
The action of time reversal on the derivative of the field is a bit more tricky. Since

$$\begin{aligned}[\partial_\mu\phi](\vec{x}, t) &= \int d\Omega_{\vec{k}}(-ik_\mu)\left[a(\vec{k})e^{-ikx} - b^\dagger(\vec{k})e^{ikx}\right], \\ [\partial_\mu\phi](\vec{x}, -t) &= \int d\Omega_{\vec{k}}(-ik_\mu)\left[a(\vec{k})e^{i\eta^{\mu\mu}k_\mu x^\mu} - b^\dagger(\vec{k})e^{-i\eta^{\mu\mu}k_\mu x^\mu}\right],\end{aligned}$$

recalling that T is an anti-linear operator, one has

$$\begin{aligned}T[\partial_\mu\phi](\vec{x}, t)T &= \int d\Omega_{\vec{k}}(ik_\mu)\left[Ta(\vec{k})Te^{ikx} - Tb^\dagger(\vec{k})Te^{-ikx}\right] \\ &= \eta_T \int d\Omega_{\vec{k}}(ik_\mu)\left[a(-\vec{k})e^{ikx} - b^\dagger(-\vec{k})e^{-ikx}\right] \\ &= \eta_T \int d\Omega_{\vec{k}}(i\eta^{\mu\mu}k_\mu)\left[a(\vec{k})e^{i\eta^{\mu\mu}k_\mu x^\mu} - b^\dagger(\vec{k})e^{-i\eta^{\mu\mu}k_\mu x^\mu}\right] \\ &= -\eta_T \eta^{\mu\mu}[\partial_\mu\phi](\vec{x}, -t),\end{aligned}$$

where in the third step we have exchanged as usual $\vec{k} \to -\vec{k}$. Thus one can compute the action of time reversal on the current:

$$\begin{aligned}
TJ_\mu(\vec{x},t)T &= T\{i\phi^\dagger(\vec{x},t)[\partial_\mu\phi](\vec{x},t) - i[\partial_\mu\phi^\dagger](\vec{x},t)\phi(\vec{x},t)\}T \\
&= -iT\phi^\dagger(\vec{x},t)TT[\partial_\mu\phi](\vec{x},t)T + iT[\partial_\mu\phi^\dagger](\vec{x},t)TT\phi(\vec{x},t)T \\
&= -i\eta_T\phi^\dagger(\vec{x},-t)(-\eta_T\eta^{\mu\mu})[\partial_\mu\phi](\vec{x},-t) + i(-\eta_T\eta^{\mu\mu})[\partial_\mu\phi^\dagger](\vec{x},-t)\eta_T\phi(\vec{x},-t), \\
&= \eta^{\mu\mu}J_\mu(\vec{x},-t),
\end{aligned}$$

Where in the last step it has been used the property $\eta_T^2 = 1$.
This result, combined with the transformation properties of the field A^μ (Exercise 1), causes the term $A^\mu J_\mu$, appearing in the scalar QED Lagrangian, to transform as $TA^\mu J_\mu T = -\eta_T A^\mu J_\mu$ so that, by choosing $\eta_T = -1$, it is possible to make this coupling time reversal-invariant.

Exercise 3

We prove the invariance of the free Dirac theory under parity.
The action is written in terms of Weyl spinors as

$$S = \int d^4x\, i\chi_L(x)\bar{\sigma}^\mu\partial_\mu\chi_L(x) + i\chi_R(x)\sigma^\mu\partial_\mu\chi_R(x) - m(\chi_R^\dagger(x)\chi_L(x) + \chi_L^\dagger(x)\chi_R(x))$$

where under parity

$$U_P^\dagger\chi_L(x)U_P = \eta_R\chi_R(x_P)$$
$$U_P^\dagger\chi_R(x)U_P = \eta_L\chi_L(x_P)$$

with $x_P^\mu = \mathscr{P}^\mu{}_\nu x^\nu$. We show that in order for the action to be parity invariant it must hold: $\eta_R = \eta_L$. At first, let us note

$$U_P^\dagger\partial_\mu\chi_L(x)U_P = \eta_R\mathscr{P}\mu^{\ \nu}\partial_\nu\chi_R(x_P)$$
$$U_P^\dagger\partial_\mu\chi_R(x)U_P = \eta_L\mathscr{P}\mu^{\ \nu}\partial_\nu\chi_L(x_P)$$
$$\mathscr{P}^\mu{}_\nu\bar{\sigma}^\nu = \sigma^\mu$$

From these relations, it follows

$$\begin{aligned}
U_P^\dagger\, S\, U_P &= \int d^4x\, U_P^\dagger\Big(i\chi_L(x)\bar{\sigma}^\mu\partial_\mu\chi_L(x) + i\chi_R(x)\sigma^\mu\partial_\mu\chi_R(x) - m(\chi_R^\dagger(x)\chi_L(x) + \chi_L^\dagger(x)\chi_R(x))\Big)U_P \\
&= \int d^4x\ \ \eta_R^2 i\chi_R(x_P)\bar{\sigma}^\nu\mathscr{P}\nu^{\ \mu}\partial\mu\chi_R(x_P) + \eta_L^2 i\chi_L(x_P)\sigma^\nu\mathscr{P}\nu^{\ \mu}\partial\mu\chi_L(x_P) \\
&\qquad\qquad - \eta_R\eta_L m(\chi_L^\dagger(x_P)\chi_R(x_P) + \chi_R^\dagger(x_P)\chi_L(x_P)) \\
&= \int d^4x\ \ \eta_R^2 i\chi_R(x)\sigma^\mu\partial_\mu\chi_R(x) + \eta_L^2 i\chi_L(x)\bar{\sigma}^\mu\partial_\mu\chi_L(x) - \eta_R\eta_L m(\chi_L^\dagger(x)\chi_R(x) + \chi_R^\dagger(x)\chi_L(x))
\end{aligned}$$

Therefore, the theory is invariant if $\eta_R = \eta_L = \eta_P$.
We can now recast the Weyl spinors into a Dirac spinor and study how it transforms under parity. Indeed

$$\Psi(x) = \begin{pmatrix}\chi_L(x)\\ \chi_R(x)\end{pmatrix}$$

And from transformations law of Weyl spinors we get

$$U_P^\dagger\Psi(x)U_P = \eta_P\begin{pmatrix}\chi_R(x_P)\\ \chi_L(x_P)\end{pmatrix} = \eta_P\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\chi_L(x_P)\\ \chi_R(x_P)\end{pmatrix} = \eta_P\gamma^0\Psi(x_P)$$

We are now ready to study how bilinears of Dirac fields transform under parity.

We want to compute the transformation properties of all the bilinears of the form $\bar{\psi}\Gamma\psi$, where Γ is some 4×4 matrix. In order to do this, it is sufficient to compute the transformation properties for

$$\Gamma = \{1_4, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \gamma^{\mu\nu}\},$$

since we have proved that any 4×4 matrix can be decomposed into a linear combination of these quantities. Before proceeding further, it is useful to work out some properties of the gamma matrices: in particular we want to find a close form for $(\gamma^\mu)^\dagger$ and $(\gamma^\mu)^T$. Let's start from the latter. Recalling the expression for the gamma matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

we can immediately see that the only hermitian matrix is γ^0 while the other ones are anti-hermitian. This suggests the following formula:

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0.$$

Indeed, making use of the Clifford algebra of the gamma matrices, $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, we get

$$\gamma^0\gamma^\mu\gamma^0 = \begin{cases} (\gamma^0)^3 = \gamma^0, & \mu = 0, \\ \gamma^0\gamma^i\gamma^0 = -\gamma^i(\gamma^0)^2 = -\gamma^i, & \mu = i = 1,2,3, \end{cases}$$

which is $= (\gamma^\mu)^\dagger$.
Similarly one can guess a formula for $(\gamma^\mu)^T$: γ^0, γ^2 are symmetric while γ^1, γ^3 are antisymmetric. Hence

$$\gamma^0\gamma^2\gamma^\mu\gamma^2\gamma^0 = \begin{cases} \gamma^0\gamma^2\gamma^0\gamma^2\gamma^0 = -(\gamma^2)^2\gamma^0 = \gamma^0 & \mu = 0, \\ \gamma^0(\gamma^2)^3\gamma^0 = -\gamma^0\gamma^2\gamma^0 = \gamma^2 & \mu = 2, \\ \gamma^0\gamma^2\gamma^i\gamma^2\gamma^0 = \gamma^0\gamma^2\gamma^2\gamma^0\gamma^i = -\gamma^i & \mu = i = 1,3, \end{cases}$$

which is $= (\gamma^\mu)^T$.

Now we are able to compute the transformation properties of the bilinears:

$$P\bar{\psi}(t,\vec{x})\Gamma\psi(t,\vec{x})P = \eta_P^2\bar{\psi}(t,-\vec{x})\gamma^0\Gamma\gamma^0\psi(t,-\vec{x}) \equiv \bar{\psi}\gamma^0\Gamma\gamma^0\psi.$$

From now on we won't write the argument of the field and it will be understood that the parity transformation changes the sign of the spatial coordinates.

Everything is now reduced to understanding what $\gamma^0\Gamma\gamma^0$ is. Let's see it case by case:

- Let's start from the simplest case: $\Gamma = 1_4$. Then:

  $$\gamma^0\gamma^0 = 1\,.$$

  This means that:

  $$P\bar{\psi}\psi P = \bar{\psi}\psi.$$

  Due to its transformation under parity, this object is called a *scalar*.

- Let us consider now $\Gamma = \gamma^5$. Hence

  $$\gamma^0\gamma^5\gamma^0 = -\gamma^5\,.$$

  This means that:

  $$P\bar{\psi}\gamma^5\psi P = -\bar{\psi}\gamma^5\psi.$$

  Due to its transformation under parity, this object is called a *pseudo-scalar*.

- The next one is $\Gamma = \gamma^\mu$:

  $$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger = \eta^{\mu\mu}\gamma^\mu.$$

  This means that

  $$P\bar{\psi}\gamma^\mu\psi P = \eta^{\mu\mu}\bar{\psi}\gamma^\mu\psi.$$

  Due to its transformation under parity, this object is called a *vector*.

- The following term is $\Gamma = \gamma^\mu \gamma^5$:

  $$\gamma^0 \gamma^\mu \gamma^5 \gamma^0 = -\eta^{\mu\mu} \gamma^\mu \gamma^5.$$

  This means that

  $$P \bar{\psi} \gamma^\mu \gamma^5 \psi P = -\eta^{\mu\mu} \bar{\psi} \gamma^\mu \gamma^5 \psi.$$

  Due to its transformation under parity, this object is called a *pseudo-vector*.

- The last term is $\Gamma = \gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$:

  $$\gamma^0 \gamma^{\mu\nu} \gamma^0 = \eta^{\mu\mu} \eta^{\nu\nu} \gamma^{\mu\nu}.$$

  This means that

  $$P \bar{\psi} \gamma^{\mu\nu} \psi P = \eta^{\mu\mu} \eta^{\nu\nu} \bar{\psi} \gamma^{\mu\nu} \psi.$$

  Due to its transformation under parity, this object is called a tensor.