Quantum Field Theory II

Set 2: solutions

Exercise 1 (Optional): The Casimirs of the Poincaré group

We start with

$$J^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \tag{1}$$

meaning $\vec{E} = (E_x, E_y, E_z)$, $\vec{B} = (B_x, B_y.B_z)$. Now, we can perform a series of rotation and boosts to get $\vec{E} = (E, 0, 0)$ and $\vec{B} = (B, 0, 0)$:

- First, we can make $\vec{E}$ and $\vec{B}$ parallel by performing a boost in the $\vec{E} \times \vec{B}$ direction. (Note that if E and B are perpendicular, we can boost to make $\vec{B}$ vanish).
- Then, we can perform a rotation to bring E and B to $\vec{E} = (E, 0, 0)$, $\vec{B} = (B, 0, 0)$.

Now we already have $J^{\mu\nu}$ in the desired form.

In the case of P^μ, starting with,

$$P^\mu = \begin{pmatrix} P^0 \\ P^1 \\ P^2 \\ P^3 \end{pmatrix}, \tag{2}$$

we can perform a rotation in the y, z plane to make $P_3 = 0$. Then, we can boost in the x direction (note that this won't affect E and B) to make $P_1 = 0$.

$$P^\mu = \begin{pmatrix} P^0 \\ 0 \\ P^2 \\ 0 \end{pmatrix} \tag{3}$$

Now we exhausted Lorentz transformation to get a minimal set of components. This constructions shows that there are 4 quantities that are invariant under Lorentz transformations. Writing them in a covariant way:

$$C_1 = J^{\mu\nu} J_{\mu\nu} \quad C_2 = \epsilon_{\mu\nu\rho\sigma} J^{\mu\nu} J^{\rho\sigma} \quad C_3 = P^\mu P_\mu \quad C_4 = W^\mu W_\mu \tag{4}$$

In an analogy with electrodynamics,

$$C_1 = E^2 - B^2, \quad C_2 = E \cdot B, \quad C_3 = (P^0)^2 - (P^2)^2, \quad C_4 = g_{\mu\alpha} \left(\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma \right) \left(\frac{1}{2} \epsilon^{\alpha\beta\epsilon\delta} J_{\beta\epsilon} P_\delta \right), \tag{5}$$

we see that that C_1, C_2, C_3 and C_4 contain as much information as E, B, P^0, P^2 but have the advantage of being manifestly Lorentz invariant.

Lastly since $[J^{\mu\nu} J_{\mu\nu}, P_\rho] \neq 0$ and $\epsilon_{\mu\nu\rho\sigma}[J^{\mu\nu} J^{\rho\sigma}, P^\alpha] \neq 0$, C_1 and C_2 cannot be Casimirs. And, as we showed in the previous exercise sheet, C_3 and C_4 commute with all the Poincaré generators. With this, we conclude that $C_3 = P^\mu P_\mu$ and $C_4 = W^\mu W_\mu$ are the two Casimirs of the Poincaré group.

Exercise 2: Maxwell's equations and transverse components

We will now show that Maxwell's equations describe indeed only two dynamical degrees of freedom. We recall that the Bianchi identity $\epsilon_{\mu\nu\rho\sigma}\partial^\mu F^{\rho\sigma} = 0$ translates into the two homogenous Maxwell equations once the field strength is written in terms of the electric and magnetic field:

$$\vec{\nabla} \wedge \vec{E} + \dot{\vec{B}} = 0\,, \qquad \vec{\nabla} \cdot \vec{B} = 0.$$

The above equations are trivially solved once we express $\vec{E}$ and $\vec{B}$ in terms of the four potential A_μ. On the other hand, we can let $\vec{E} = \vec{E}_L + \vec{E}_\perp$, and similarly for $\vec{B}$, with:

$$E^i_\perp \equiv \left(\delta^{ij} - \frac{\partial^i \partial^j}{\nabla^2}\right) E^j\,, \qquad E^i_L \equiv E^i - E^i_\perp\,, \qquad \partial^i E^i_\perp = 0.$$

Then the above equations become:

$$\vec{\nabla} \wedge \vec{E}_\perp = -\dot{\vec{B}}_\perp\,, \qquad \vec{\nabla} \cdot \vec{B}_\perp = 0,$$
$$0 = \vec{\nabla} \wedge \vec{E}_L = -\dot{\vec{B}}_L\,, \qquad \vec{\nabla} \cdot \vec{B}_L = 0.$$

The longitudinal component of the magnetic field is constant in time, and has divergence, curl and thus laplacian equal to 0. Hence it is a constant. Since the field has to vanish at infinity, it can only be identically 0, thus:

$$\vec{B}_L = 0\,, \qquad \vec{\nabla} \wedge \vec{E}_\perp = -\dot{\vec{B}}_\perp.$$

The evolution of the magnetic field is completely determined by the one of the electric field and so only 3 degrees of freedom are present in $F^{\mu\nu}$.
The other two Maxwell equations are $\partial_\mu F^{\mu\nu} = J^\nu$, where we have allowed for a current coupled to the electromagnetic field. In terms of $\vec{E}, \vec{B}$ they read:

$$\vec{\nabla} \cdot \vec{E} = \rho\,, \qquad \vec{\nabla} \wedge \vec{B} - \dot{\vec{E}} = \vec{J}.$$

Again decomposing in longitudinal and transverse components we get:

$$\vec{\nabla} \cdot \vec{E}_L = \rho,$$

which fixes the longitudinal part of the electric field completely. Hence the only dynamical component is $\vec{E}_\perp$, which contains only 2 degrees of freedom.

Let us now see how this can be obtained using the four potential A_μ. The inhomogeneous Maxwell equations split in the following way (recall $\partial_\mu = (\partial_0, \vec{\nabla})$, $A_\mu = (A_0, -\vec{A})$):

$$\nu = 0: \qquad -\nabla^2 A_0 - \vec{\nabla} \cdot \dot{\vec{A}} = \vec{\nabla} \cdot \vec{E} = J^0,$$
$$\nu = i: \qquad \ddot{\vec{A}} + \vec{\nabla} \dot{A}_0 - \nabla^2 \vec{A} + \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) = \vec{\nabla} \wedge \vec{B} - \dot{\vec{E}} = \vec{J}.$$

The solution of the first equation is formally:

$$A_0 = -\nabla^{-2}(J^0 + \vec{\nabla} \cdot \dot{\vec{A}}).$$

Plugging this into the second equation, we get:

$$\ddot{A}^i - \partial_i \nabla^{-2}(\dot{J}^0 + \vec{\nabla} \cdot \ddot{\vec{A}}) - \nabla^2 A^i + \partial_i(\partial_j A^j) = J^i$$
$$\Longrightarrow \left(\delta^{ij} - \frac{\partial^i \partial^j}{\nabla^2}\right) \ddot{A}^j - \nabla^2 \left(\delta^{ij} - \frac{\partial^i \partial^j}{\nabla^2}\right) A^j = J^i + \partial_i \nabla^{-2} \dot{J}^0.$$

We can recognize a wave equation for the projected component $A^i_\perp = (\delta^{ij} - \frac{\partial^i \partial^j}{\nabla^2})A^j$. The combination in parenthesis is indeed a projector since, if squared, it is equal to itself. In momentum space we can see that it projects A^i on the direction orthogonal to the momentum p:

$$A^i_\perp = \left(\delta^{ij} - \frac{p^i p^j}{p^2}\right) A^j \implies p^i A^i_\perp = 0.$$

This is equivalent to imposing the Coulomb condition (in that gauge $\vec{\nabla} \cdot \vec{A}_L = \vec{\nabla} \wedge \vec{A}_L = 0$ thus $A_L = 0$). Indeed the Coulomb gauge identifies the physical degrees of freedom, even if it is not a Lorentz invariant constraint. Moreover the *longitudinal* part $A^i_L = A^i - A^i_\perp$ does not appear anywhere: it is completely decoupled.

Exercise 3: Coulomb gauge quantization

Let us consider the Lagrangian for a massless vector field interacting with a source J^μ

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu \ .$$

This can be rewritten explictly in terms of the spatial and time components of $A_\mu = (A_0, -\vec{A})$:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu = -2\times\frac{1}{4}F_{0i}F^{0i} - \frac{1}{4}F_{ij}F^{ij} - J^iA_i - J^0A_0 \\ &= \frac{1}{2}\left(\dot{\vec{A}} + \vec{\nabla}A_0\right)^2 - \frac{1}{2}\left(\vec{\nabla}\wedge\vec{A}\right)^2 - J^0A_0 + \vec{J}\cdot\vec{A},\end{aligned}$$

where we have used:

$$\left(\vec{\nabla}\wedge\vec{A}\right)^k = \left(\partial_iA^j - \partial_jA^i\right)\epsilon^{ijk} \qquad \Longrightarrow \qquad \left(\vec{\nabla}\wedge\vec{A}\right)^2 = \frac{1}{2}\left(\partial_iA^j - \partial_jA^i\right)\left(\partial_iA^j - \partial_jA^i\right).$$

Supposing that the matter current J^μ is conserved, this Lagrangian is known to be invariant under gauge transformations:

$$A_\mu \longrightarrow A_\mu - \partial_\mu\Lambda,$$

where Λ is an arbitrary function. In order to deal with the gauge freedom, it is useful to impose a condition on A_μ. There are various gauges which have been found useful in many applications, for instance:

- Lorentz gauge: $\partial_\mu A^\mu = 0$;
- Coulomb gauge: $\vec{\nabla}\cdot\vec{A} = 0$;
- Temporal gauge: $A^0 = 0$;
- Axial gauge: $A^3 = 0$.

We are interested in studying quantization in Coulomb gauge.

In general $\vec{\nabla}\cdot\vec{A} \neq 0$. However it is possible to consider the (equivalent) gauge transformed field $A'_\mu = A_\mu - \partial_\mu\Lambda$; with a suitable choice of Λ we can impose :

$$0 = \vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \nabla^2\Lambda = 0 \qquad \Longleftrightarrow \qquad \nabla^2\Lambda = -\vec{\nabla}\cdot\vec{A}.$$

This is just Poisson equation with a source $-\vec{\nabla}\cdot\vec{A}$, which can always be solved with the use of the Green function:

$$\Lambda(\vec{x},t) = -\left[\nabla^{-2}(\vec{\nabla}\cdot\vec{A})\right](\vec{x},t) = \int\frac{d^3y}{4\pi}\frac{\vec{\nabla}\cdot\vec{A}(\vec{y},t)}{|\vec{x}-\vec{y}|}.$$

Therefore we can always choose a configuration where, dropping the prime, $\vec{\nabla}\cdot\vec{A} = 0$.

Let us compute the conjugate momenta of the fields A_μ. In principle we expect four momenta Π^μ:

$$\begin{aligned}\Pi^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_tA_\mu)} = -F^{\alpha\beta}\frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_tA_\mu)} = -F^{0\mu}, \\ \Pi^0 &= 0, \qquad\qquad \Pi^i = -F^{0i} = \partial^iA^0 - \partial^0A^i = \partial_tA_i - \partial_iA_0.\end{aligned}$$

The conjugate momentum of the field A_0 is identically vanishing. This of course cannot be compatible with the canonical commutation relation $[A^\mu(\vec{x},t), \Pi_\nu(\vec{y},t)] = i\delta^\mu_\nu\delta^3(\vec{x}-\vec{y})$. This suggests that the zero component of the vector potential is not a dynamical variable, even if it appears explicitly in the Lagrangian. This can be seen also looking at the zero component of the equations of motion:

$$J^0 = \partial_iF^{i0} = \partial_i\Pi^i = -\vec{\nabla}\cdot\dot{\vec{A}} - \nabla^2A_0 \qquad \Longrightarrow \qquad -\nabla^2A_0(\vec{x},t) = J^0(\vec{x},t).$$

We used the Coulomb gauge condition to write the last expression. The above equation is just Poisson equation for the field A_0 with source given $J^0(\vec{x},t)$. This can be solved to give the usual Coulomb potential:

$$A_0(\vec{x},t) = \int \frac{d^3y}{4\pi} \frac{J^0(\vec{y},t)}{|\vec{x}-\vec{y}|}.$$

The time component of A_μ is then automatically determined at any time by the value of the charge density J^0 at that time. This shows that neither A_0 nor Π_0 are dynamical variables[1].

We are left with six canonical variables: $(A^i, \Pi_i) \equiv (\vec{A}, \vec{\Pi})$. However these are still subject to constraints. The first constraint is just Coulomb gauge condition $\vec{\nabla}\cdot\vec{A} = 0$. The second one arises from the equation of motion $\partial_i F^{i0} = J^0$ written in terms of $\Pi_i = -F_{0i}$:

$$-\partial^i \Pi_i = \vec{\nabla}\cdot\vec{\Pi} = J^0.$$

Once we try to quantize this theory we face with the problem of defining consistently with this constraint the canonical commutation relations. It is indeed immediate to realize that a commutation relation of the form $[A(\vec{y},t)^i, \Pi(\vec{x},t)_j] = i\delta^i_j \delta^3(\vec{x}-\vec{y})$ has non vanishing divergence, hence it is not consistent with the constraints. A choice of the commutation relations which is consistent with the two constraints $\vec{\nabla}\cdot\vec{\Pi} = J^0$ and $\vec{\nabla}\cdot\vec{A} = 0$ is the following[2]:

$$[A(\vec{x},t)^i, \Pi(\vec{y},t)_j] = i\delta^i_j \delta^3(\vec{x}-\vec{y}) - i\partial^{(x)}_i \partial^{(y)}_j \frac{1}{4\pi|\vec{x}-\vec{y}|} = i\delta^i_j \delta^3(\vec{x}-\vec{y}) - i\frac{\partial^{(x)}_i \partial^{(x)}_j}{\vec{\nabla}^2_x}\delta^3(\vec{x}-\vec{y}).$$

We can easily check the consistency. Taking the derivative $\partial^{(x)}_i$ of both sides of the relation, from the l.h.s. we must get zero, from the r.h.s. we have

$$i\partial^{(x)}_j \delta^3(\vec{x}-\vec{y}) - i\partial^{(y)}_j \underbrace{\nabla^2 \frac{1}{4\pi|\vec{x}-\vec{y}|}}_{-\delta^3(\vec{x}-\vec{y})} = i\partial^{(x)}_j \delta^3(\vec{x}-\vec{y}) - i\partial^{(x)}_j \delta^3(\vec{x}-\vec{y}) = 0.$$

Exacly the same computation shows that we get zero when we derive with respect to $\partial^{(y)}_j$ on the r.h.s. On the l.h.s. we get:

$$\partial^{(y)}_j [A(\vec{x},t)^i, \Pi(\vec{y},t)_j] = [A(\vec{x},t)^i, \vec{\nabla}\cdot\vec{\Pi}(\vec{y},t)_j] = [A(\vec{x},t)^i, J^0(\vec{y},t)_j] = 0,$$

since J^0 depends only on the source degrees of freedom and hence commutes with the field A_μ. Hence it is consistent to use this commutation relation to quantize the theory.

The final needed step is to write the Hamiltonian density. Recalling:

$$\vec{\Pi} = \vec{\nabla}A^0 + \dot{\vec{A}}.$$

we easily find

$$\mathcal{H} = \dot{\vec{A}}\cdot\vec{\Pi} - \mathcal{L} = \left(\vec{\Pi} - \vec{\nabla}A_0\right)\cdot\vec{\Pi} - \frac{1}{2}\vec{\Pi}^2 + \frac{1}{2}\left(\vec{\nabla}\wedge\vec{A}\right)^2 + J^0 A_0 - \vec{J}\cdot\vec{A}.$$

Since A_0 is fixed by the charge density and it is not dynamical, it is natural to remove its contribution to the momentum as

$$\vec{\Pi}_\perp = \vec{\Pi} - \vec{\nabla}A_0.$$

Then the Hamiltonian takes the form:

$$\begin{aligned}\mathcal{H} &= \vec{\Pi}_\perp \cdot \left(\vec{\Pi}_\perp + \vec{\nabla}A_0\right) - \frac{1}{2}\left(\vec{\Pi}_\perp + \vec{\nabla}A_0\right)^2 + \frac{1}{2}\left(\vec{\nabla}\wedge\vec{A}\right)^2 + J^0 A_0 - \vec{J}\cdot\vec{A} \\ &= \frac{1}{2}\vec{\Pi}^2_\perp + \frac{1}{2}\left(\vec{\nabla}\wedge\vec{A}\right)^2 + \left[J^0 A_0 - \frac{1}{2}(\vec{\nabla}A_0)^2\right] - \vec{J}\cdot\vec{A}.\end{aligned}$$

[1] Notice that indeed in the Lagrangian no time derivative of A_0 appears.

[2] A formal justification of the choice of the commutation relation can be achieved through Dirac theory of Hamiltonian constraint, see sect.a 7.6, 8.2 and 8.3 of Weinberg book 1.

The term in square parenthesis can be rewritten as a Coulomb interaction. Indeed using $-\vec{\nabla}^2 A_0 = J^0$, we get

$$\begin{aligned} J^0(\vec{x},t)A_0(\vec{x},t) - \frac{1}{2}(\vec{\nabla}A_0(\vec{x},t))^2 &= J^0(\vec{x},t)A_0(\vec{x},t) - \frac{1}{2}\vec{\nabla}\left(A_0(\vec{x},t)\,\vec{\nabla}A_0(\vec{x},t)\right) + \frac{1}{2}A_0(\vec{x},t)\,\vec{\nabla}^2 A_0(\vec{x},t) \\ = \frac{1}{2}J^0(\vec{x},t)A_0(\vec{x},t) - \frac{1}{2}\vec{\nabla}\left(A_0(\vec{x},t)\,\vec{\nabla}A_0(\vec{x},t)\right) &= \frac{1}{2}\int d^3x \frac{J^0(\vec{x},t)J^0(\vec{y},t)}{4\pi|\vec{x}-\vec{y}|} + \text{"total derivative"}. \end{aligned} \tag{6}$$

The reader can easily convince himself that the quantization procedure is consistent looking at the equation of motion for the $U(1)$ field. Considering for simplicity the case whith no source, $J^\mu = 0$, so that $A_0 = 0$, one finds:

$$\begin{aligned} i\dot{A}^i(\vec{x},t) &= \int d^3y\,[A^i(\vec{x},t), \mathcal{H}(\vec{y},t)] = \int d^3y\,[A^i(\vec{x},t), \Pi_j(\vec{y},t)]\Pi_j(\vec{y},t) \\ &= \int d^3y\, i\left(\delta^i_j \delta^3(\vec{x}-\vec{y}) - i\partial_i^{(x)}\partial_j^{(y)}\frac{1}{4\pi|\vec{x}-\vec{y}|}\right)\Pi_j(\vec{y},t). \end{aligned} \tag{7}$$

Replacing $\partial_i^{(x)}$ by $-\partial_i^{(y)}$, integrating by parts and using $\vec{\nabla}\cdot\vec{\Pi} = J^0 = 0$, we get:

$$\dot{\vec{A}} = \vec{\Pi}. \tag{8}$$

Similarly, one can show:

$$i\dot{\vec{\Pi}}(\vec{x},t) = i\ddot{\vec{A}}(\vec{x},t) = \int d^3y\,[\vec{\Pi}(\vec{x},t), \mathcal{H}(\vec{y},t)] \quad\Longrightarrow\quad \Box\vec{A} = 0. \tag{9}$$

These are the expected equations of motion for the canonical variables. One can thus proceed expanding $\vec{A}$ and $\vec{\Pi}$ in terms of creation and annihilation operators in the standard way[3].

Exercise 4: Energy momentum tensor

Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\lambda}{2}(\partial_\rho A^\rho)^2.$$

The equations of motion are:

$$\partial_\mu F^{\mu\nu} + \lambda\partial^\nu(\partial_\rho A^\rho) = \Box A^\nu - (1-\lambda)\partial^\nu(\partial_\rho A^\rho) = 0.$$

The energy momentum tensor can be derived using the usual procedure:

$$\begin{aligned} &x'^\mu = x^\mu - a^\mu, \\ &A'_\rho(x') = A_\rho(x) \simeq A_\rho(x') + a^\nu \underbrace{\partial_\nu A_\rho(x')}_{\Delta_{\rho\nu}}. \end{aligned}$$

Thus we get:

$$J^\mu_i = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\Delta_{a\,i} - \epsilon^\mu_i\mathcal{L} \implies T^\mu_{\ \nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\rho)}\Delta_{\rho\,\nu} - \delta^\mu_\nu\mathcal{L}.$$

And hence:

$$T^\mu_{\ \nu} = -F^{\mu\rho}\partial_\nu A_\rho + \frac{1}{4}\delta^\mu_\nu F_{\alpha\beta}F^{\alpha\beta} - \lambda(\partial_\alpha A^\alpha)\partial_\nu A^\mu + \frac{\lambda}{2}\delta^\mu_\nu(\partial_\rho A^\rho)^2.$$

We can now check explicitly, that the divergence of the energy momentum tensor vanishes as predicted by Noether's theorem:

$$\begin{aligned} \partial_\mu T^\mu_{\ \nu} = &-\partial_\mu F^{\mu\rho}\partial_\nu A_\rho - F^{\mu\rho}\partial_\mu\partial_\nu A_\rho + \frac{1}{2}F^{\alpha\beta}\partial_\nu F_{\alpha\beta} \\ &- \lambda(\partial_\alpha A^\alpha)\partial_\nu(\partial_\mu A^\mu) - \lambda\partial_\mu(\partial_\alpha A^\alpha)\partial_\nu A^\mu + \lambda(\partial_\alpha A^\alpha)\partial_\nu(\partial_\mu A^\mu). \end{aligned}$$

[3] Notice that the constraints $\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\vec{\Pi} = 0$ reduce by one the number of polarizations, from three to two; for details see any book, e.g. Maggiore sec. 4.3.1.

Using the equation of motion for the first term, we can eliminate all the terms proportional to λ. What remains is only:

$$-F^{\mu\rho}\partial_\mu\partial_\nu A_\rho + \frac{1}{2}F^{\alpha\beta}\partial_\nu F_{\alpha\beta} = -\frac{1}{2}F^{\mu\rho}(\partial_\mu\partial_\nu A_\rho - \partial_\rho\partial_\nu A_\mu) + \frac{1}{2}F^{\alpha\beta}\partial_\nu F_{\alpha\beta},$$

where we have used the antisymmetry of the field strength. Expanding this gives:

$$-\frac{1}{2}F^{\alpha\beta}\left(\partial_\alpha\partial_\nu A_\beta - \partial_\beta\partial_\nu A_\alpha - \partial_\nu F_{\alpha\beta}\right) = -\frac{1}{2}F^{\alpha\beta}\left(\partial_\alpha\partial_\nu A_\beta - \partial_\beta\partial_\nu A_\alpha - \partial_\nu\partial_\alpha A_\beta + \partial_\nu\partial_\beta A_\alpha\right) = 0.$$

In the limit $\lambda \to 0$, the Lagrangian is gauge invariant. Therefore, it could be expected that also the energy momentum tensor is gauge invariant but this is in fact not true, since the Noether formula contains a derivative with respect to the field A_μ which is not gauge invariant (only the tensor $F^{\mu\nu}$ is). Instead under a gauge transformation:

$$T^\mu_{\ \nu} \longrightarrow T^\mu_{\ \nu} + F^{\mu\rho}\partial_\nu\partial_\rho\Lambda.$$

On the contrary however, the charges, which are related to physical quantities, must be gauge invariant and indeed they are:

$$P_\nu = \int d^3x\ T^0_{\ \nu} \longrightarrow \int d^3x\ T^0_{\ \nu} + \int d^3x\ F^{0i}\partial_\nu\partial_i\Lambda = \int d^3x\ T^0_{\ \nu} - \int d^3x\ \partial_i F^{0i}\partial_\nu\Lambda = \int d^3x\ T^0_{\ \nu},$$

where we have integrated by parts and used the equation of motion $\partial_i F^{i0} = 0$. To summarize, although the Noether procedure gives us a non gauge invariant energy momentum tensor, the charges are invariant.
One can always modify the definition of the energy momentum tensor by adding a piece $K^\mu_{\ \nu}$ which is divergenceless and such that $K^0_{\ \nu}$ is a total space derivative. In the present case we can define:

$$\tilde{T}^\mu_{\ \nu} = T^\mu_{\ \nu} + F^{\mu\rho}\partial_\rho A_\nu = F^{\mu\rho}F_{\rho\nu} + \frac{1}{4}\delta^\nu_\mu F_{\alpha\beta}F^{\alpha\beta}.$$

Indeed

$$\partial_\mu(F^{\mu\rho}\partial_\rho A_\nu) = 0\ , \qquad F^{0\rho}\partial_\rho A_\nu = F^{0i}\partial_i A_\nu = \partial_i(F^{0i}A_\nu) + \text{eq. of motion.}$$

Since $\tilde{T}^\mu_{\ \nu}$ depends only on the field strength, it is gauge invariant. This could be argued also by noticing that the gauge variation of $F^{\mu\rho}\partial_\rho A_\nu$ exactly compensates the one of $T^\mu_{\ \nu}$. Notice finally that $\tilde{T}^\mu_{\ \nu}$ is symmetric and traceless:

$$\tilde{T}^\mu_{\ \mu} = F^{\mu\rho}F_{\rho\mu} + \frac{1}{4}4F_{\alpha\beta}F^{\alpha\beta} = 0.$$

When $\lambda \neq 0$, differentiating the equations of motion with respect to x^ν gives:

$$\partial_\nu\partial_\mu F^{\mu\nu} + \lambda\Box(\partial_\rho A^\rho) \equiv \lambda\Box(\partial_\rho A^\rho) = 0,$$

where the first piece is equal to 0 due to the antisymmetry of the $F^{\mu\nu}$. Since $\Box(\partial_\rho A^\rho) = 0$, if $\partial_\rho A^\rho = 0$ and $\partial_t(\partial_\rho A^\rho) = 0$ at *a particular time*, it is true that $\partial_\rho A^\rho = 0$ identically. So if both these conditions are satisfied and $\partial_\rho A^\rho = 0$, then quantities computed with the equations of motion (like the Noether currents and charges) do not depend on λ. But, in general, if $\partial_\rho A^\rho \neq 0$, a λ-dependence remains.