Quantum Field Theory II

Set 1: solutions

Exercise 1: The Pauli–Lubanski (pseudo)vector

- We have

  $$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}[J_{\nu\rho}, P_\sigma] + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\sigma J_{\nu\rho} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}[J_{\nu\rho}, P_\sigma] + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu J_{\rho\sigma}, \tag{1}$$

  where in the last equality we relabeled $\sigma \leftrightarrow \nu$ and used anti-symmetry of $\epsilon^{\mu\nu\rho\sigma}$ and $J_{\rho\sigma}$.

  Now all we have to show is that the term involving the commutator is zero. It follows from the algebra (eq. (3.191) in the lecture notes),

  $$[J_{\nu\rho}, P_\sigma] = i(\eta_{\sigma\rho}P_\nu - \eta_{\sigma\nu}P_\rho), \tag{2}$$

  that

  $$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}[J_{\nu\rho}, P_\sigma] = \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}(\eta_{\sigma\rho}P_\nu - \eta_{\sigma\nu}P_\rho) = 0. \tag{3}$$

  The first term is zero because $\eta_{\sigma\rho}$ is symmetric and its indices are contracted with $\epsilon^{\mu\nu\rho\sigma}$ which is anti-symmetric. The second term is zero for the same reason.

- 1. It follows from the algebra (eq. (3.192) from the lectures notes) and anti-symmetry of the Levi-Civita tensor that

     $$W^\mu P_\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma P_\mu = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}[P_\sigma, P_\mu] = 0. \tag{4}$$

  2. We have

     $$[P^\mu, W^\nu] = \frac{1}{2}\epsilon^{\nu\alpha\rho\sigma}[P^\mu, J_{\alpha\beta}P_\sigma] = \frac{1}{2}\epsilon^{\nu\alpha\rho\sigma}([P^\mu, J_{\alpha\beta}]P_\sigma + J_{\alpha\beta}[P^\mu, P_\sigma]). \tag{5}$$

     Now we make use of the algebra, in particular $[P^\mu, P_\sigma] = 0$ and eq. (2) to find

     $$[P^\mu, W^\nu] = \frac{1}{2}\epsilon^{\nu\alpha\rho\sigma}([P^\mu, J_{\alpha\rho}]P_\sigma = \frac{i}{2}\epsilon^{\nu\alpha\rho\sigma}(\delta^\mu_\alpha P_\rho - \delta^\mu_\rho P_\alpha)P_\sigma = 0, \tag{6}$$

     due to antisymmetry of the Levi-Civita tensor.

  3. We have

     $$[J^{\mu\nu}, W^\rho] = \frac{1}{2}\epsilon^{\rho\alpha\beta\sigma}[J^{\mu\nu}, J_{\alpha\beta}P_\sigma] = \frac{1}{2}\epsilon^{\rho\alpha\beta\sigma}([J^{\mu\nu}, J_{\alpha\beta}]P_\sigma + J_{\alpha\beta}[J^{\mu\nu}, P_\sigma]). \tag{7}$$

     We again make use of the algebra,

     $$[J^{\mu\nu}, J_{\alpha\beta}] = i(\delta^\nu_\alpha J^\mu{}_\beta - \delta^\mu_\alpha J^\nu{}_\beta + \delta^\mu_\beta J^\nu{}_\alpha - \delta^\nu_\beta J^\mu{}_\alpha), \qquad [J^{\mu\nu}, P_\sigma] = i(\delta^\nu_\sigma P^\mu - \delta^\mu_\sigma P^\nu), \tag{8}$$

     to get

     $$[J^{\mu\nu}, W^\rho] = \frac{i}{2}(\epsilon^{\rho\nu\beta\sigma}J^\mu{}_\beta P_\sigma + \epsilon^{\rho\alpha\nu\sigma}J_\alpha{}^\mu P_\sigma + \epsilon^{\rho\alpha\beta\nu}J_{\alpha\beta}P_\mu) - (\mu \leftrightarrow \nu). \tag{9}$$

     We now factor out $J_{\alpha\beta}P_\sigma$ to have

     $$[J^{\mu\nu}, W^\rho] = \frac{i}{2}J_{\alpha\beta}P_\sigma(\eta^{\mu\alpha}\epsilon^{\rho\nu\beta\sigma} + \eta^{\mu\beta}\epsilon^{\rho\alpha\nu\sigma} + \eta^{\mu\sigma}\epsilon^{\rho\alpha\beta\nu}) - (\mu \leftrightarrow \nu). \tag{10}$$

     We now make use of the identity[1]

     $$\eta^{\mu\nu}\epsilon^{\rho\alpha\beta\sigma} = \eta^{\mu\rho}\epsilon^{\nu\alpha\beta\sigma} + \eta^{\mu\alpha}\epsilon^{\rho\nu\beta\sigma} + \eta^{\mu\beta}\epsilon^{\rho\alpha\nu\sigma} + \eta^{\mu\sigma}\epsilon^{\rho\alpha\beta\nu} \tag{12}$$

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[1] This identity follows from expanding the determinant relation

$$\det(A)\epsilon^{\rho\alpha\beta\sigma} = A^\rho{}_{\rho'}A^\alpha{}_{\alpha'}A^\beta{}_{\beta'}A^\sigma{}_{\sigma'}\epsilon^{\rho'\alpha'\beta'\sigma'} \tag{11}$$

to linear order in $\omega^\rho{}_{\rho'}$, where $A^\rho{}_{\rho'} = \delta^\rho_{\rho'} + \omega^\rho{}_{\rho'} + O(\omega^2)$. The left-hand-side becomes proportional to the trace $\mathrm{tr}\,\omega = \eta^{\mu\nu}\omega_{\mu\nu}$, matching to the left-hand-side of (12). Correpondingly, the right-hand-sides will match and given that $\omega^{\mu\nu}$ is arbitrary, the identity (12) will hold.

to have

$$[J^{\mu\nu}, W^\rho] = \frac{i}{2} J_{\alpha\beta} P_\sigma (\eta^{\mu\nu}\epsilon^{\rho\alpha\beta\sigma} - \eta^{\mu\rho}\epsilon^{\nu\alpha\beta\sigma}) - (\mu \leftrightarrow \nu) = \tag{13}$$

$$= -\frac{i}{2} J_{\alpha\beta} P_\sigma \eta^{\mu\rho}\epsilon^{\nu\alpha\beta\sigma} + \frac{i}{2} J_{\alpha\beta} P_\sigma \eta^{\nu\rho}\epsilon^{\mu\alpha\beta\sigma} = i(\eta^{\nu\rho}W^\mu - \eta^{\mu\rho}W^\nu). \tag{14}$$

Note that replacing W^ρ with the momentum P^ρ in the above yields the correct commutation relation between $J^{\alpha\beta}$ and P^ρ.This result is a consequence of the fact that W^ρ transforms as a 4-vector under Lorentz transformations (just like P^ρ).

4. We have

$$[W^\mu, W^\nu] = \frac{1}{2}\epsilon^{\nu\alpha\rho\sigma}[W^\mu, J_{\alpha\beta}P_\sigma] = \frac{1}{2}\epsilon^{\nu\alpha\rho\sigma}([W^\mu, J_{\alpha\beta}]P_\sigma + J_{\alpha\beta}[W^\mu, P_\sigma]). \tag{15}$$

We now make use of the results found in points 2 and 3. We find

$$[W^\mu, W^\nu] = \frac{i}{2}\epsilon^{\nu\alpha\rho\sigma}(\delta^\mu_\alpha W_\rho - \delta^\mu_\rho W_\alpha)P_\sigma = i\epsilon^{\nu\alpha\rho\sigma}W_\rho P_\sigma. \tag{16}$$

- To show that W^2 is a Casimir we have to show that it commutes with the Poincare generators P^μ and $J^{\mu\nu}$. We have

$$[W^2, P^\mu] = W_\alpha[W^\alpha, P^\mu] + [W^\alpha, P^\mu]W_\alpha = 0 \tag{17}$$

due to the result derived in point 2.

We also have

$$[W^2, J^{\mu\nu}] = W_\alpha[W^\alpha, J^{\mu\nu}] + [W^\alpha, J^{\mu\nu}]W_\alpha = i[W^\mu, W^\nu] + i[W^\nu, W^\mu] = 0, \tag{18}$$

where we made use of the result derived in point 3.

Exercise 2: Spin of Dirac Fermion states

We want to compute the action of the angular momentum operator computed in exercise 14.2:

$$J_k = \int d^3x\, \psi^\dagger_\alpha(t,\vec{x}) \left([\vec{x} \wedge (-i\vec{\nabla}_x)]_k \delta_{\alpha\beta} + \frac{(\Sigma_k)_{\alpha\beta}}{2} \right) \psi_\beta(t,\vec{x})\,.$$

Here the spinor indices have been made explicit but are dropped in the following in favor of matrix notation. Looking at this operator, one might be tempted to identify the two terms as giving angular momentum and spin:

$$\vec{L} \stackrel{?}{=} \int d^3x\, \psi^\dagger(t,\vec{x})[\vec{x} \wedge (-i\vec{\nabla}_x)]\psi(t,\vec{x})\ , \qquad \vec{S} \stackrel{?}{=} \int d^3x\, \psi^\dagger(t,\vec{x})\frac{(\Sigma_k)}{2}\psi(t,\vec{x})\,. \tag{19}$$

But, as we will see below, this is not true. Indeed the physical states with a definite spin or angular momentum are not eigenstates of these operators. Instead, the contributions of these two terms will mix. One should remember that physically, the orbital angular momentum and the spin are not separately conserved, only their sum is conserved.

To exemplify this, we consider a generic single-particle (one electron) state defined as

$$|\Psi\rangle = \int d^3x\, f_\alpha(x)\psi^\dagger_\alpha(x)|0\rangle\,.$$

Using the mode decomposition of the field, we can express the state in the basis of momentum-polarization states

$$\begin{aligned}|\Psi\rangle &= \int d^3x \sum_r \int d\Omega_p\, u^\dagger_\alpha(r,p)e^{ip\cdot x}f_\alpha(x)|p,r\rangle \\ &= \sum_r \int d\Omega_p\, F(r,p)|p,r\rangle\,,\end{aligned}$$

where we defined

$$F(r,p) = u^\dagger_\alpha(r,p)\tilde{f}_\alpha(p)\,,$$

with

$$\tilde{f}_\alpha(p) = \int d^3x f_\alpha(x) e^{ip\cdot x}\,.$$

Our goal is to derive the action of $\vec{J}$ on the wavefunction $F(p,r)$, and split it into two contributions, one of which acts on the momentum only while the other affets the polarization index r only. The fact that the naive splitting (19) does not work is a consequence of (20) below.

The computation goes as follows. First let's compute

$$\begin{aligned}
[\vec{J},\psi^\dagger_\alpha(x)] &= \int d^3y \Big[\psi^\dagger(t,\vec{y})\Big(\vec{y}\wedge(-i\vec{\nabla}_y)+\vec{\Sigma}/2\Big)\psi(t,\vec{y}),\psi^\dagger_\alpha(t,\vec{x})\Big]\\
&= \int d^3y\ \psi^\dagger(t,\vec{y})\Big(\vec{y}\wedge(-i\vec{\nabla}_y)+\vec{\Sigma}/2\Big)\Big\{\psi(t,\vec{y}),\psi^\dagger_\alpha(t,\vec{x})\Big\}\\
&= \Big[\psi^\dagger(t,\vec{x})\Big(\vec{x}\wedge(i\overleftarrow{\nabla}_x)+\vec{\Sigma}/2\Big)\Big]_\alpha\,,
\end{aligned}$$

where the left-arrow notation means

$$\psi^\dagger(t,\vec{x})\big(\vec{x}\wedge(i\overleftarrow{\nabla}_x)\big) = i\vec{x}\wedge\vec{\nabla}_x\,\psi^\dagger(t,\vec{x})\,.$$

In the second line we have used $[AB,C]=A\{B,C\}-\{A,C\}B$ and $\{\psi^\dagger_\alpha(t,\vec{y}),\psi^\dagger_\beta(t,\vec{x})\}=0$. In the third line we used $\{\psi_\alpha(t,\vec{y}),\psi^\dagger_\beta(t,\vec{x})\}=\delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y})$ and performed the integral over $\vec{y}$.

Now we can apply $\vec{J}$ on the state. A sequence of elementary manipulations yields.

$$\begin{aligned}
\vec{J}|\Psi\rangle &= \int d^3x f_\alpha(x)[\vec{J},\psi^\dagger_\alpha(x)]|0\rangle\\
&= \int d^3x f_\alpha(x)\Big[\psi^\dagger(t,\vec{x})\Big(\vec{x}\wedge(i\overleftarrow{\nabla}_x)+\vec{\Sigma}/2\Big)\Big]_\alpha|0\rangle\\
&= \int d^3x f_\alpha(x)\sum_r\int d\Omega_p\Big[e^{ip\cdot x}u^\dagger(r,p)\Big(\vec{x}\wedge(i\overleftarrow{\nabla}_x)+\vec{\Sigma}/2\Big)\Big]_\alpha|p,r\rangle\\
&= \int d^3x f_\alpha(x)\sum_r\int d\Omega_p\Big[u^\dagger(r,p)\Big(-\vec{x}\wedge\vec{p}+\vec{\Sigma}/2\Big)e^{ip\cdot x}\Big]_\alpha|p,r\rangle\\
&= \int d^3x f_\alpha(x)\sum_r\int d\Omega_p\Big[u^\dagger(r,p)\Big(-i\vec{p}\wedge\vec{\partial}_p+\vec{\Sigma}/2\Big)e^{ip\cdot x}\Big]_\alpha|p,r\rangle\\
&= \sum_r\int d\Omega_p\Big[u^\dagger(r,p)\Big(-i\vec{p}\wedge\vec{\partial}_p+\vec{\Sigma}/2\Big)\Big]_\alpha\tilde{f}_\alpha(p)|p,r\rangle\,.
\end{aligned}$$

It can be checked that

$$\begin{aligned}
\frac{\vec{\Sigma}}{2}u(r,p) &= i(\vec{p}\wedge\vec{\partial}_p)u(r,p)+\sum_{r'}\frac{(\vec{\sigma})_{r'r}}{2}u(r',p)\,, \qquad (20)\\
u^\dagger(r,p)\frac{\vec{\Sigma}}{2} &= (-i\vec{p}\wedge\vec{\partial}_p)u^\dagger(r,p)+\sum_{r'}\frac{(\vec{\sigma})_{rr'}}{2}u^\dagger(r',p)\,.
\end{aligned}$$

so that we can rewrite the result as

$$\begin{aligned}
\vec{J}|\Psi\rangle &= \sum_{r,r'}\int d\Omega_p\Big((-i\vec{p}\wedge\vec{\partial}_p)\delta_{rr'}+(\vec{\sigma})_{rr'}/2\Big)\Big(u^\dagger_\alpha(r',p)\tilde{f}_\alpha(p)\Big)|p,r\rangle\\
&= \sum_{r,r'}\int d\Omega_p\Big((-i\vec{p}\wedge\vec{\partial}_p)\delta_{rr'}+(\vec{\sigma})_{rr'}/2\Big)F(r',p)\,|p,r\rangle\,.
\end{aligned}$$

To summarize, we have found that angular momentum acts as

$$F(r,p)\xrightarrow{\vec{J}}\sum_{r'}\Big((-i\vec{p}\wedge\vec{\partial}_p)\delta_{rr'}+(\vec{\sigma})_{rr'}/2\Big)F(r',p)\,.$$

In this equation describing the transformation of the wavefunction, we can now identify the first term as orbital angular momentum, since it acts only on the momentum dependence. The second term is spin, since it acts only on polarization indices.

For example, to get states of zero angular momentum, we consider states such that $(-i\vec{p}\wedge\vec{\partial}_p)F(r,p) = 0$. An s-wave state, which has $F(r,p) = F(r,|\vec{p}|)$ has this property. In this case, the action of J_3 is simply

$$J_3|\Psi\rangle = S_3|\Psi\rangle = \int d\Omega_p \left(\frac{1}{2}F(1,p)|p,1\rangle - \frac{1}{2}F(2,p)|p,2\rangle\right).$$

Moreover

$$\begin{cases} F(2,p) = 0 \Rightarrow \quad S_3|\Psi\rangle = \frac{1}{2}|\Psi\rangle, \\ F(1,p) = 0 \Rightarrow \quad S_3|\Psi\rangle = -\frac{1}{2}|\Psi\rangle. \end{cases}$$

In the first case, we have a state of spin $+\frac{1}{2}$ and in the second case, spin $-\frac{1}{2}$.

Exercise 3: Transformation of the boost generators under translations

As it has been shown in the lecture,

$$g(\Lambda_1, a_1)g(\Lambda_2, a_2) = g(\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1), \tag{21}$$

with $g(\Lambda, a) \in ISO(3,1)$ implies,

$$\begin{aligned} g(\Lambda, a) &= g(\mathbb{1}, a)g(\Lambda, 0), \\ g^{-1}(\Lambda, a) &= g(\Lambda^{-1}, -\Lambda^{-1}a). \end{aligned} \tag{22}$$

With this,

$$g(\mathbb{1}, -a)g(\Lambda, 0)g(\mathbb{1}, a) = g(\Lambda, -a)g(\mathbb{1}, a) = g(\Lambda, \Lambda a - a). \tag{23}$$

Now, if we expand $\Lambda = \mathbb{1} + \omega$ with $\omega = -\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}$, we have,

$$g(\mathbb{1}, -a)g(\Lambda, 0)g(\mathbb{1}, a) = e^{-ia\cdot P}\left(\mathbb{1} - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)e^{ia\cdot P} = \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}e^{-ia\cdot P}J^{\mu\nu}e^{ia\cdot P}, \tag{24}$$

also,

$$g(\Lambda, \Lambda a - a) = g(1+\omega, \omega a) = \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} + i\omega_{\mu\nu}a^\nu P^\mu = \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} + \frac{i}{2}\omega_{\mu\nu}\left(a^\nu P^\mu - a^\mu P^\nu\right) \tag{25}$$

where in the last step we have used $\omega_{\mu\nu} = -\omega_{\nu\mu}$. This antisymmetrization is important because $J^{\mu\nu}$ is antisymmetric. Comparing both expressions above we get the desired result:

$$e^{-iaP}J^{\mu\nu}e^{iaP} = J^{\mu\nu} + P^\nu a^\mu - P^\mu a^\nu. \tag{26}$$