Quantum Field Theory

Homework 2: solutions

Exercise 1: two real scalar fields

In order to find the physical spectrum, one has to diagonalize the kinetic term. In order to do so, define:

$$\begin{cases} \phi_1 = \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2) \\ \phi_2 = \frac{1}{\sqrt{2}}(\varphi_1 - \varphi_2) \end{cases} \quad \text{or equivalently} \quad \begin{cases} \varphi_1 = \frac{1}{\sqrt{2}}(\phi_1 + \phi_2) \\ \varphi_2 = \frac{1}{\sqrt{2}}(\phi_1 - \phi_2) \end{cases} \tag{1}$$

The different terms in the Lagrangian become

$$\begin{aligned} \frac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 + \frac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2 \to & \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 \\ g\partial_\mu\varphi_1\partial^\mu\varphi_2 \to & \frac{g}{2}(\partial_\mu\phi_1\partial^\mu\phi_1 - \partial_\mu\phi_2\partial^\mu\phi_2) \\ (\varphi_1^2 + \varphi_2^2) \to & (\phi_1^2 + \phi_2^2) \end{aligned}$$

So :

$$\mathcal{L} = \frac{1}{2}(1+g)\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}(1-g)\partial_\mu\phi_2\partial^\mu\phi_2 + \frac{m^2}{2}(\phi_1^2 + \phi_2^2) + \frac{\lambda}{4!}(\phi_1^2 + \phi_2^2)^2 \tag{2}$$

In order to have a physically acceptable theory, we need both kinetic terms to be positive (see homework 1), so :

$$|g| < 1 \tag{3}$$

In order to find the physical masses, we canonically normalize the fields:

$$\Phi_1 = \frac{\phi_1}{\sqrt{1+g}} \quad \text{and} \quad \Phi_2 = \frac{\phi_2}{\sqrt{1-g}} \tag{4}$$

and we find :

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi_1\partial^\mu\Phi_1 + \frac{1}{2}\partial_\mu\Phi_2\partial^\mu\Phi_2 + \frac{m^2\Phi_1^2}{2(1+g)} + \frac{m^2\Phi_2^2}{2(1-g)} + \frac{\lambda\Phi_1^4}{4!(1-g)^2} + \frac{\lambda\Phi_2^4}{4!(1-g)^2} + \frac{2\lambda\Phi_1^2\Phi_2^2}{4!(1-g^2)} \tag{5}$$

So we have two physical particles of masses $m_1^2 = \frac{m^2}{1+g}$ and $m_2^2 = \frac{m^2}{1-g}$.

Note : The case $g = \pm 1$ is more subtle because the theory becomes strogly coupled and will not be treated here. However, when $\lambda = 0$, then we are left with (in the $g = 1$ case) :

$$\mathcal{L} = \partial_\mu\phi_1\partial^\mu\phi_1 + \frac{m^2}{2}(\phi_1^2 + \phi_2^2) \tag{6}$$

ϕ_2 has no kinetic term and is called an auxiliary field. We can solve explicitly the equations of motion and get : $\phi_2 = 0$. We are left with one physical particle of mass $\frac{m}{\sqrt{2}}$.

Exercise 2: Fermi Lagrangian

Given the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu + \bar{\psi}(i\partial\!\!\!/ - qA\!\!\!/)\psi,$$

the equations of motion for the vector field are:

$$-\partial_\mu(\partial^\mu A^\rho - \partial^\rho A^\mu) = M^2 A^\rho - J^\rho,$$

where $J^\rho \equiv q\bar{\psi}\gamma^\rho\psi$. In Fourier space they read:

$$[(k^2 - M^2)g^{\mu\rho} - k^\mu k^\rho]\tilde{A}_\mu(k) = -\tilde{J}^\rho(k).$$

Expanding for $k \ll M$, we get:

$$\tilde{A}_\mu(k) \simeq \frac{1}{M^2}\tilde{J}_\mu(k) \quad \Longrightarrow \quad A_\mu(x) \simeq \frac{1}{M^2}J_\mu(x) = \frac{q}{M^2}\bar{\psi}(x)\gamma_\mu\psi(x).$$

Note that the same result can be obtained by solving the equation of motion for the field A_μ without any approximation, and then taking the low energy limit of the solution. In this case we consider the Green's function $G_{\sigma\alpha}(x)$, satisfying the defining equation:

$$[-(\partial_\mu\partial^\mu + M^2)g^{\rho\sigma} + \partial^\rho\partial^\sigma]G_{\sigma\alpha}(x) = \delta^\rho_\alpha\delta^{(4)}(x).$$

To find the explicit form of the Green's function it is convenient to work in Fourier space, where the equation becomes $[(k^2 - M^2)g^{\rho\sigma} - k^\rho k^\sigma]\tilde{G}_{\sigma\alpha}(k) = \delta^\rho_\alpha$. Looking for a solution of the form $\tilde{G}_{\sigma\alpha}(k) = Ak_\sigma k_\alpha + Bg_{\sigma\alpha}$ (the only two tensor structures available), we get in the end:

$$\tilde{G}_{\sigma\alpha}(k) = \frac{1}{k^2 - M^2}\left(g_{\sigma\alpha} - \frac{k_\sigma k_\alpha}{M^2}\right).$$

The solution for the field A_μ is then given by the convolution of $G_{\sigma\alpha}(x)$ with J^α:

$$A_\mu(x) = -\int d^4y \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - M^2}\left(g_{\mu\alpha} - \frac{k_\mu k_\alpha}{M^2}\right)e^{-ik(x-y)}J^\alpha(y).$$

In the low energy limit $k \ll M$ we obtain:

$$A_\mu(x) \simeq \int d^4y \int \frac{d^4k}{(2\pi)^4}\frac{g_{\mu\alpha}}{M^2}e^{-ik(x-y)}J^\alpha(y) = \frac{J_\mu(x)}{M^2}.$$

Plugging this result in the equation of motion for the field ψ, namely $(i\partial\!\!\!/ - qA\!\!\!/)\psi = 0$, we find:

$$\left(i\partial^\mu - \frac{q^2}{M^2}\bar{\psi}\gamma^\mu\psi\right)\gamma_\mu\psi = 0,$$

which can be interpreted as derived from a *Fermi* effective Lagrangian:

$$\mathcal{L}_F = \bar{\psi}i\partial\!\!\!/\psi - \frac{q^2}{2M^2}\bar{\psi}\gamma^\mu\psi\ \bar{\psi}\gamma_\mu\psi.$$

Exercise 3: Parity and Spinor Representation

With the definition of the spinor field in the usual way,

$$\Psi_D(x) \equiv \begin{pmatrix}\psi_L(x)\\ \psi_R(x)\end{pmatrix}, \tag{7}$$

where $\psi_L = (j_-, j_+)$ and $\psi_R = (j_+, j_-)$. Choosing, for example, the left-handed representation of the Lorentz group:

$$\Lambda_L(\vec{\theta}, \vec{\eta}) \tag{8}$$

Since a representation of parity changes the sign of the boost generators, we have:

$$P\Lambda_L(\vec{\theta}, \vec{\eta})P = \Lambda_L(\vec{\theta}, -\vec{\eta}) \tag{9}$$

But the last quantity is nothing but the right-handed representation, so that:

$$P\Lambda_{L/R}(\vec{\theta}, \vec{\eta})P = \Lambda_{R/L}(\vec{\theta}, \vec{\eta}) \tag{10}$$

Under a boost,

$$\begin{aligned} \psi_L(x) &\to \Lambda_L(\theta, \eta)\psi_L(\Lambda^{-1}x), \\ \psi_R(x) &\to \Lambda_R(\theta, \eta)\psi_R(\Lambda^{-1}x) = \Lambda_L(\theta, -\eta)\psi_R(\Lambda^{-1}x). \end{aligned} \tag{11}$$

With this and using Eq. (10) we can conclude that under parity our spinor transforms,

$$U_P \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} U_P = \eta \begin{pmatrix} \psi_R(x_P) \\ \psi_L(x_P) \end{pmatrix}, \tag{12}$$

Now particularizing for $(1/2, 0)$, as parity exchanges $(1/2, 0)$ and $(0, 1/2)$, it is realized on $\Psi_D(x)$ as

$$P : \Psi_D(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \to \Psi_D^P(x) \equiv \begin{pmatrix} \psi_R(Px) \\ \psi_L(Px) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_L(Px) \\ \psi_R(Px) \end{pmatrix} \equiv \gamma^0 \Psi_D(Px) \tag{13}$$

The matrix γ^0 swaps ψ_L and ψ_R and represents parity on the spinorial indices. Indeed, one has $\gamma_0^2 = 1_{4\times4}$ (remember $P^2 = 1$). Moreover, as Lorentz transformations are represented on Ψ_D by the block diagonal matrices

$$\Lambda_D \equiv \begin{pmatrix} \Lambda_L(\theta, \eta) & 0 \\ 0 & \Lambda_R(\theta, \eta) \end{pmatrix}, \tag{14}$$

we have

$$\gamma^0 \Lambda_D(\vec{\theta}, \vec{\eta}) = \Lambda_D(\vec{\theta}, -\vec{\eta})\gamma^0. \tag{15}$$