Quantum Field Theory

Set 9: solutions

Exercise 1

Let us start with the 2-body decay: the differential decay width is

$$d\Gamma_{A\to CD} = \frac{1}{2E_A}|\mathcal{M}_{A\to CD}|^2 d\Phi_2.$$

In the center of mass the energy of the particle A is simply given by its mass, that we call M. The 2-body phase space can be computed exactly as the one computed for the $2\to 2$ scattering, integrating in d^3p_D first and then transforming dp_C into $d(E_C+E_D)$. The final result is identical in both cases where here we must substitute $\sqrt{s}$ with M:

$$d\Gamma_{A\to CD} = \frac{1}{2M}|\mathcal{M}_{A\to CD}|^2 \frac{d\varphi\; d\cos\theta}{16\pi^2}\frac{p_C(M)}{M}.$$

The 3-body phase space instead is more complicated. Before proceeding, let us review the kinematics of a 3-body decay in the center of mass. A particle with four momentum $P_A = (M,\vec{0})$ decays into three particles with four momenta P_B, P_C, P_D. Momentum conservation reads

$$P_A = P_B + P_C + P_D \quad\Longrightarrow\quad \begin{cases} M = E_B + E_C + E_D, \\ \vec{0} = \vec{p}_B + \vec{p}_C + \vec{p}_D. \end{cases}$$

These are 4 conditions, hence we are left with 5 degrees of freedom. We can understand this geometrically. Since the three momenta add up to the null vector, they are coplanar. Let $\hat{n}$ be the unit vector normal to this plane. We can choose two of the independent variables as the two angles that define the direction of $\hat{n}$. The triangle defined by the three momenta can rotate rigidly in the plane; we take the angle that defines the orientation of the triangle as the third variable. Finally the last two variables define the shape and the size of the triangle. We can always choose two energies E_B and E_C to parametrize the latter. If we are not interested in the polarisation of the initial state the dependence on the angles of the matrix element is irrelevant and we can describe the final state with the last two variables only.
In practice we can parametrize the directions of the three vectors identifying the $\hat{x}$ axis with the direction of $\vec{p}_B$ and calling θ_C and θ_D the angles of the others vectors with respect to it. We can now introduce the Mandelstam invariants

$$s = (P_A - P_B)^2 = (P_C+P_D)^2, \qquad t = (P_A - P_C)^2 = (P_B + P_D)^2, \qquad u = (P_A - P_D)^2 = (P_B + P_C)^2.$$

One can easily show that these variables are not independent: they are related by momentum conservation

$$0 = (P_A - P_B - P_C - P_D)^2 \quad\Longrightarrow\quad M^2 + m_B^2 + m_C^2 + m_D^2 = s + t + u.$$

Hence there are only two independent invariant quantities that we can build starting from the momenta. Their expressions in the center of mass are

$$\begin{aligned} s &= M^2 + m_B^2 - 2ME_B, \\ t &= M^2 + m_C^2 - 2ME_C = m_B^2 + m_D^2 + 2(E_BE_D - p_Bp_D\cos\theta_D), \\ u &= M^2 + m_D^2 - 2ME_D = m_B^2 + m_C^2 + 2(E_BE_C - p_Bp_C\cos\theta_C). \end{aligned}$$

Let's now come back to the decay amplitude. In the center of mass, it reads

$$\begin{aligned} d\Gamma_{A\to BCD} &= \frac{1}{2E_A}|\mathcal{M}_{A\to BCD}|^2 d\Phi_3 \\ &= \frac{1}{2M}|\mathcal{M}_{A\to BCD}|^2 \frac{d^3p_B}{(2\pi)^3 2E_B}\,\frac{d^3p_C}{(2\pi)^3 2E_C}\,\frac{d^3p_D}{(2\pi)^3 2E_D}\,(2\pi)^4\delta^4\left(P_A - P_B - P_C - P_D\right). \end{aligned}$$

Let's first integrate over d^3p_D and eliminate the δ^3 of spatial momentum conservation. In addition let us use polar coordinates for the other momenta:

$$d\Gamma_{A\to BCD} = \frac{1}{2M}|\mathcal{M}_{A\to BCD}|^2 \frac{p_B^2 dp_B\, p_C^2 dp_C}{E_C E_B E_D}\, \frac{d\varphi_C\, d\cos\theta_C\, d\Omega_B}{8(2\pi)^5}\, \delta\left(M - E_B - E_C - E_D\right),$$

where again $d\Omega_B$ stands for the solid angle. Notice that the integration on d^3p_D has set

$$E_D = \sqrt{m_D^2 + p_D^2} = \sqrt{m_D^2 + (\vec{p}_B + \vec{p}_C)^2} = \sqrt{m_D^2 + p_B^2 + p_C^2 + 2p_C p_B \cos\theta_C} \quad .$$

Therefore we can substitute in the decay amplitude

$$d\cos\theta_C = dE_D \frac{E_D}{p_C p_B},$$

togheter with $dp_B = \frac{E_B}{p_B} dE_B$ and $dp_C = \frac{E_C}{p_C} dE_C$ in order to get

$$\begin{aligned} d\Gamma_{A\to BCD} &= \frac{1}{2M}|\mathcal{M}_{A\to BCD}|^2 dE_B\, dE_C\, dE_D \frac{d\varphi_C\, d\Omega_B}{8(2\pi)^5}\, \delta\left(M - E_B - E_C - E_D\right) \\ &= \frac{1}{2M}|\mathcal{M}_{A\to BCD}|^2 dE_B\, dE_C \frac{d\varphi_C\, d\Omega_B}{8(2\pi)^5}, \end{aligned}$$

where in the last step we have integrated in dE_D. Notice the presence of 5 degrees of freedom in this process. In order to proceed further we make an **assumption** on the structure of the matrix element $\mathcal{M}_{A\to BCD}$, namely we assume that it depends only on the Mandelstam variables s, t, u. This is always true in the following two cases:

- A, B, C, and D are scalar particles
- A, B, C, or D are not scalar particles but we sum over the polarizations of the final state and we average on the polarizations of the initial state.

Under these assumptions, the decay is invariant under rotations, so no dependence on the angles is there in $|\mathcal{M}_{A\to BCD}|^2$, so that the effective number of degrees of freedom is two. Finally:

$$d\Gamma_{A\to BCD} = \frac{1}{2M}|\mathcal{M}_{A\to BCD}|^2 \frac{dE_B dE_C}{4(2\pi)^3}.$$

This is our final expression: we cannot go further without knowing the explicit form of the Matrix element. Sometimes one is interested in the decay amplitude as a function of different variables. for example one can express easily what we have found in terms of two independent Mandelstam invariants, say s and t. The only thing we need is the Jacobian of the change of variables:

$$dE_B dE_C = \frac{ds dt}{|J|}, \qquad J = \det\begin{pmatrix} \frac{\partial s}{\partial E_B} & \frac{\partial s}{\partial E_C} \\ \frac{\partial t}{\partial E_B} & \frac{\partial t}{\partial E_C} \end{pmatrix} = \det\begin{pmatrix} -2M & 0 \\ 0 & -2M \end{pmatrix} = 4M^2.$$

Hence

$$d\Gamma_{A\to BCD} = \frac{1}{32M^3}|\mathcal{M}_{A\to BCD}|^2 \frac{ds\, dt}{(2\pi)^3}.$$

Exercise 2

Let us consider the free scalar propagator for real scalar fields defined as

$$\mathcal{D}(x-y) \equiv \langle 0|\, T(\phi(x)\phi(y))\, |0\rangle\,,$$

where the time ordered product is defined by

$$T(\phi(x)\phi(y)) \equiv \theta(x_0 - y_0)\phi(x)\phi(y) + \theta(y_0 - x_0)\phi(y)\phi(x).$$

We want to find an expression for this propagator. The strategy is the following: we will show that this quantity obeys a differential equation which can be easily solved in Fourier space. Once we have the solution in momentum space we need to anti-transform to coordinate space. Let us start considering the following quantity:

$$(\Box_x + m^2)\mathcal{D}(x-y) = \left(\frac{\partial^2}{\partial x_0^2} - \nabla^2 + m^2\right)\mathcal{D}(x-y).$$

Let us consider the time derivatives first:

$$\begin{aligned}
&\frac{\partial^2}{\partial x_0^2}\left(\theta(x_0-y_0)\phi(x)\phi(y) + \theta(y_0-x_0)\phi(y)\phi(x)\right) \\
&= \frac{\partial}{\partial x_0}\left(\delta(x_0-y_0)\left[\phi(x)\,,\,\phi(y)\right] + \theta(x_0-y_0)\partial_0\phi(x)\phi(y) + \theta(y_0-x_0)\phi(y)\partial_0\phi(x)\right) \\
&= \delta'(x_0-y_0)\left[\phi(x)\,,\,\phi(y)\right] + 2\delta(x_0-y_0)\left[\partial_0\phi(x)\,,\,\phi(y)\right] + \theta(x_0-y_0)\partial_0^2\phi(x)\phi(y) + \theta(y_0-x_0)\phi(y)\partial_0^2\phi(x).
\end{aligned}$$

Adding the space derivative with respect to x and the mass we get

$$\begin{aligned}
&(\Box_x + m^2)T(\phi(x)\phi(y)) = \\
&= \delta'(x_0-y_0)\left[\phi(x)\,,\,\phi(y)\right] + 2\delta(x_0-y_0)\left[\partial_0\phi(x)\,,\,\phi(y)\right] + \\
&\theta(x_0-y_0)(\Box_x+m^2)\phi(x)\phi(y) + \theta(y_0-x_0)\phi(y)(\Box_x+m^2)\phi(x),
\end{aligned}$$

and the last two terms vanish since the field ϕ satisfies the Klein-Gordon equation $(\Box + m^2)\phi = 0$.
Let us manipulate further the above equation. This expression has to be thought of as a distributions. This means that it makes sense only inside an integral and in addition we can make use of identities which are true once this expression is evaluated inside an integral. In order to see how this work in practice let us introduce an integral $\int dx_0$ and a test function $f(x_0)$. Hence we can write:

$$\begin{aligned}
&\int dx_0 f(x_0)\left\{\partial_0\delta(x_0-y_0)\left[\phi(x_0,\vec{x})\,,\,\phi(y_0,\vec{y})\right] + 2\delta(x_0-y_0)\left[\partial_0\phi(x_0,\vec{x})\,,\,\phi(y_0,\vec{y})\right]\right\} = \\
&\int dx_0\left\{-f(x_0)\delta(x_0-y_0)\left[\partial_0\phi(x_0,\vec{x})\,,\,\phi(y_0,\vec{y})\right] - \partial_0 f(x_0)\delta(x_0-y_0)\left[\phi(x_0,\vec{x})\,,\,\phi(y_0,\vec{y})\right] + \right. \\
&\quad \left. 2f(x_0)\delta(x_0-y_0)\left[\partial_0\phi(x_0,\vec{x})\,,\,\phi(y_0,\vec{y})\right]\right\} = \\
&\int dx_0\left\{-\partial_0 f(x_0)\delta(x_0-y_0)\left[\phi(x_0,\vec{x})\,,\,\phi(x_0,\vec{y})\right] + f(x_0)\delta(x_0-y_0)\left[\partial_0\phi(x_0,\vec{x})\,,\,\phi(x_0,\vec{y})\right]\right\},
\end{aligned}$$

where we have integrated by parts in the first step ($f(x_0)$ can be taken fast decreasing at infinity) and we have replaced $y_0 \to x_0$ in the last equality since, because of the presence of the $\delta(x_0-y_0)$, the value of the integral doesn't change. Now that we have seen how we can manipulate the distributions, we can eliminate the integral and write:

$$(\Box_x + m^2)T(\phi(x)\phi(y)) = \delta(x_0-y_0)\left[\partial_0\phi(x_0,\vec{x})\,,\,\phi(x_0,\vec{y})\right] - \delta(x_0-y_0)\left[\phi(x_0,\vec{x})\,,\,\phi(x_0,\vec{y})\right]\frac{\partial}{\partial x_0}.$$

The two commutators appearing in the above expression are evaluated at equal time, hence we can use the canonical commutation relations:

$$\left[\phi(x_0,\vec{x})\,,\,\phi(x_0,\vec{y})\right] = 0 \qquad\qquad \left[\partial_0\phi(x_0,\vec{x})\,,\,\phi(x_0,\vec{y})\right] = -i\delta^3(\vec{x}-\vec{y}).$$

Therefore taking the expectation value on the vacuum we get:

$$(\Box_x + m^2)\mathcal{D}(x-y) = \langle 0|\,(\Box_x+m^2)T(\phi(x)\phi(y))\,|0\rangle = -i\delta^4(x-y).$$

This is the differential equation we were looking for. We can solve this equation in momentum space: let us introduce the Fourier transform of the propagator:

$$\mathcal{D}(x-y) = \int \frac{d^4p}{(2\pi)^4}\tilde{\mathcal{D}}(p)e^{-ip(x-y)}.$$

Then in terms of $\tilde{\mathcal{D}}$ the differential equation reads

$$(-p^2+m^2)\tilde{\mathcal{D}}(p) = -i \quad\Longrightarrow\quad \tilde{\mathcal{D}}(p) = \frac{i}{p_0^2 - \vec{p}^2 - m^2}.$$

The above solution is incomplete because we must specify how the integral must be performed. Indeed substituting

$$\mathcal{D}(x-y) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \int \frac{dp_0}{(2\pi)} \frac{i}{p_0^2 - \vec{p}^2 - m^2} e^{-ip_0(x_0-y_0)}$$

we get the integral of a function with poles on the real axis: $p_0 = \pm\sqrt{\vec{p}^2 + m^2}$. We need a prescription to perform the integral on p_0. The need of preserving Lorentz invariance fixes the structure $p_0^2 - \vec{p}^2$ hence we can only add an imaginary part to the denominator:

$$\frac{i}{p_0^2 - \vec{p}^2 - m^2} \longrightarrow \tilde{\mathcal{D}}_\pm(p) = \frac{i}{p_0^2 - \vec{p}^2 - m^2 \pm i\varepsilon}.$$

Both the prescriptions define a finite integral, however they are not equivalent and we will see that only $\tilde{\mathcal{D}}_+(p)$ gives the right result. Let us compute the integral using the + prescription. We need to distinguish two cases:

- $(x_0 - y_0) > 0$: in this case the exponential $e^{-ip_0(x_0-y_0)}$ requires $\mathrm{Im}(p_0) < 0$ therefore we close the contour in the lower half-plane. The poles are shifted to $p_0 = \pm\sqrt{\vec{p}^2 + m^2 - i\varepsilon} \sim \pm\sqrt{\vec{p}^2 + m^2} \mp i\varepsilon$. There is only one pole enclosed by the contour, hence we get:

$$\begin{aligned}
\mathcal{D}(x-y) &= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \int \frac{dp_0}{(2\pi)} \frac{i}{p_0^2 - \vec{p}^2 - m^2 + i\varepsilon} e^{-ip_0(x_0-y_0)} \\
&= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \left(\frac{i}{2\pi}(-2\pi i)\mathrm{Res}\left(\frac{1}{p_0^2 - \vec{p}^2 - m^2 + i\varepsilon} e^{-ip_0(x_0-y_0)} \right)\right)\Bigg|_{p_0=\sqrt{\vec{p}^2+m^2}-i\varepsilon} \\
&= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \left(\frac{e^{-i\sqrt{\vec{p}^2+m^2}(x_0-y_0)}}{2\sqrt{\vec{p}^2+m^2}} \right) = \int d\Omega_p \, e^{-ip(x-y)}
\end{aligned}$$

- $(x_0 - y_0) < 0$: in this case the exponential $e^{-ip_0(x_0-y_0)}$ requires $\mathrm{Im}(p_0) > 0$ therefore we close the contour in the upper half-plane. Again there is only one pole enclosed by the contour, hence we get:

$$\begin{aligned}
\mathcal{D}(x-y) &= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \left(\frac{i}{2\pi}(2\pi i)\mathrm{Res}\left(\frac{1}{p_0^2 - \vec{p}^2 - m^2 + i\varepsilon} e^{-ip_0(x_0-y_0)} \right)\right)\Bigg|_{p_0=-\sqrt{\vec{p}^2+m^2}+i\varepsilon} \\
&= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \left(\frac{e^{i\sqrt{\vec{p}^2+m^2}(x_0-y_0)}}{2\sqrt{\vec{p}^2+m^2}} \right) = \int d\Omega_p \, e^{ip(x-y)}
\end{aligned}$$

  In the last step we have changed $\vec{p} \to -\vec{p}$ in the integral.

Collecting the two pieces we get:

$$\mathcal{D}(x-y) = \theta(x_0 - y_0) \int d\Omega_p \, e^{-ip(x-y)} + \theta(y_0 - x_0) \int d\Omega_p \, e^{ip(x-y)}.$$

Finally let us check that this is the right prescription. Let us compute the same quantity starting from the original definition in terms of the field ϕ and expand:

$$\phi(x) = \int d\Omega_k \left(a(x) e^{-ikx} + a^\dagger(k) e^{ikx} \right).$$

Recalling that $a(k)\left|0\right\rangle = 0$ and $\left\langle 0\right| a^\dagger(k) = 0$, we can easily obtain

$$\mathcal{D}(x-y) = \theta(x_0 - y_0) \int d\Omega_p d\Omega_q \; e^{-ipx+iqy} \left\langle 0\right| a(p)\, a^\dagger(q) \left|0\right\rangle + \theta(y_0 - x_0) \int d\Omega_p d\Omega_q \; e^{-ipy+iqx} \left\langle 0\right| a(p)\, a(q)^\dagger \left|0\right\rangle,$$

and since

$$\left\langle 0\right| a(p)\, a(q)^\dagger \left|0\right\rangle = \left\langle 0\right| \left[a(p),\, a(q)^\dagger\right] \left|0\right\rangle = (2\pi)^3 2q_0 \delta^3(\vec{p} - \vec{q}),$$

we finally obtain (integrating over d^3q)

$$\mathcal{D}(x-y) = \theta(x_0 - y_0) \int d\Omega_p \; e^{-ip(x-y)} + \theta(y_0 - x_0) \int d\Omega_p \; e^{ip(x-y)},$$

which matches the expression found before with the prescription $+i\varepsilon$. In the end the expression of the scalar propagator is

$$\mathcal{D}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{ip(x-y)}.$$

Exercise 3

In this exercise we will mimic the computation of the previous exercise in the case of a fermionic field. Again the strategy is to find a differential equation that the fermionic propagator has to satisfy and solve it in momentum space. Again, in order to come back to coordinate space we must regularize the integral. However we don't need to repeat all the steps: we already know that the right prescription is the one obtained for the scalar case.
The fermion propagator is

$$\mathcal{S}(x-y) \equiv \langle 0| \, T(\psi(x)\bar{\psi}(y)) \, |0\rangle \, ,$$

where the time ordered product is defined by

$$T(\psi(x)\bar{\psi}(y)) \equiv \theta(x_0 - y_0)\psi(x)\bar{\psi}(y) - \theta(y_0 - x_0)\bar{\psi}(y)\psi(x).$$

Notice the minus sign between the two terms in the above expression. Let us consider:

$$\begin{aligned}(i\not{\partial}_x - m)T(\psi(x)\bar{\psi}(y)) &= \left(i\gamma^0 \frac{\partial}{\partial x^0} + i\gamma^i \frac{\partial}{\partial x^i} - m\right) T(\psi(x)\bar{\psi}(y)) \\ &= i\gamma^0 \delta(x_0 - y_0) \left\{\psi(x) \, , \, \bar{\psi}(y)\right\} + \theta(x_0 - y_0)(i\not{\partial}_x - m)\psi(x)\bar{\psi}(y) - \theta(y_0 - x_0)\bar{\psi}(y)(i\not{\partial}_x - m)\psi(x).\end{aligned}$$

Making use of the Dirac equation $(i\not{\partial}_x - m)\psi(x) = 0$ we obtain:

$$(i\not{\partial}_x - m)T(\psi(x)\bar{\psi}(y)) = i\gamma^0 \delta(x_0 - y_0) \left\{\psi(x) \, , \, \bar{\psi}(y)\right\} .$$

We can substitute $y_0 \to x_0$ everywhere because of the presence of the δ-function. Hence we can use the canonical anticommutation relation

$$\left\{\psi(x_0, \vec{x}) \, , \, \bar{\psi}(y_0, \vec{y})\right\} = \gamma^0 \delta^3(\vec{x} - \vec{y})$$

to finally get:

$$(i\not{\partial}_x - m)\mathcal{S}(x-y) = \langle 0| \, (i\not{\partial}_x - m)T(\psi(x)\bar{\psi}(y)) \, |0\rangle = i\delta^4(x-y).$$

Let us introduce the Fourier transform of the fermion propagator:

$$\mathcal{S}(x-y) = \int \frac{d^4p}{(2\pi)^4} \tilde{\mathcal{S}}(p) e^{-ip(x-y)}.$$

Then the differential equation can be rewritten as:

$$(\not{p} - m)\tilde{\mathcal{S}}(p) = i.$$

The solution of the above equation is sometimes written formally as

$$\tilde{\mathcal{S}}(p) = \frac{i}{\not{p} - m}.$$

This is a short notation which stands for the inverse of the 4×4 matrix $(\not{p} - m)$. In practice one can multiply both members of the differential equation by $(\not{p} + m)$ and get:

$$i(\not{p} + m) = (\not{p} + m)(\not{p} - m)\tilde{\mathcal{S}}(p) = (\not{p}\not{p} - m^2)\tilde{\mathcal{S}}(p).$$

Using

$$\not{p}\not{p} = p_\mu p_\nu \gamma^\mu \gamma^\nu = \frac{1}{2} p_\mu p_\nu \left\{\gamma^\mu \, , \, \gamma^\nu\right\} = p^2,$$

we can invert:

$$\tilde{\mathcal{S}}(p) = \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon},$$

where we have introduced the same regularization as for the scalar propagator.

Exercise 4

Let us consider the Lagrangian for a massless vector field in an arbitrary gauge:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\left(1 - \frac{1}{\alpha}\right)(\partial_\mu A^\mu)^2.$$

For the case $\alpha = 1$ this Lagrangian reduces to the one used for the quantization à la Gupta-Bleuler. Different values of α represent different gauges. The equations of motion following from this Lagrangian are:

$$\left(\Box\delta^\mu_\nu - \left(1 - \frac{1}{\alpha}\right)\partial_\nu\partial^\mu\right) A_\mu = 0.$$

The definition of the vector propagator is the usual one :

$$\mathcal{D}_{\rho\sigma} = \langle 0|\, T(A_\rho(x)A_\sigma(y))\, |0\rangle\,,$$

where the time ordered product is

$$T(A_\rho(x)A_\sigma(y)) \equiv \theta(x_0 - y_0)A_\rho(x)A_\sigma(y) + \theta(y_0 - x_0)A_\sigma(y)A_\rho(x).$$

One can show that the above propagator is actually the inverse of the equation of motion, as in the scalar or fermionic case, and therefore it satisfies the following differential equation:

$$\left(\Box\delta^\mu_\nu - \left(1 - \frac{1}{\alpha}\right)\partial_\nu\partial^\mu\right)\mathcal{D}_{\mu\sigma} = i\eta_{\nu\sigma}\delta^4(x - y).$$

In order to solve this equation one can introduce the Fourier transform of the propagator:

$$\mathcal{D}_{\mu\nu}(x - y) = \int \frac{d^4p}{(2\pi)^4}\tilde{\mathcal{D}}_{\mu\nu}(p)e^{ip(x-y)}.$$

We get a differential equation of the form:

$$\left(p^2\delta^\mu_\nu - \left(1 - \frac{1}{\alpha}\right)p_\nu p^\mu\right)\tilde{\mathcal{D}}_{\mu\sigma}(p) = -i\eta_{\nu\sigma}$$

The expression in parentheses can be regarded as a 4×4 matrices with space-time indices, hence we need to invert such a matrix. Let us multiply by:

$$(p^2\delta^\nu_\rho - Bp_\rho p^\nu)\left(p^2\delta^\mu_\nu - \left(1 - \frac{1}{\alpha}\right)p_\nu p^\mu\right)\tilde{\mathcal{D}}_{\mu\sigma}(p) = p^2\left(p^2\delta^\mu_\rho - \left(B + 1 - \frac{1}{\alpha} - B + \frac{B}{\alpha}\right)p_\rho p^\mu\right)\tilde{\mathcal{D}}_{\mu\sigma}(p)$$

If we choose $B = (1 - \alpha)$ the term proportional to $p_\rho p^\mu$ vanishes and we get the simple expression

$$p^4\tilde{\mathcal{D}}_{\rho\sigma}(p) = -i\left(p^2\eta_{\rho\sigma} - (1 - \alpha)p_\rho p_\sigma\right).$$

Finally we can invert the propagator using the usual prescription:

$$\tilde{\mathcal{D}}_{\rho\sigma}(p) = \frac{-i}{p^2 + i\varepsilon}\left(\eta_{\rho\sigma} - (1 - \alpha)\frac{p_\rho p_\sigma}{p^2}\right).$$

The most used values of α are:

$$\alpha = 1: \ \text{Feynman gauge} \qquad \frac{-i}{p^2 + i\varepsilon}\eta_{\rho\sigma},$$

$$\alpha = 0: \ \text{Landau gauge} \qquad \frac{-i}{p^2 + i\varepsilon}\left(\eta_{\rho\sigma} - \frac{p_\rho p_\sigma}{p^2}\right).$$

The advantage of the Feynman gauge is its simplicity, while in the Landau gauge the propagator is manifestly transverse.