Quantum Field Theory

Set 8: solutions

Exercise 1: Application of Lippman-Schwinger equation

Let's first rewrite the Lippman-Schwinger equation,

$$|\psi_\alpha^+\rangle = |\phi_\alpha\rangle + \frac{1}{E_\alpha - H_0 + i\epsilon} H_I |\psi_\alpha^+\rangle,$$

in such a way as to have all the $|\psi_\alpha^+\rangle$ on the right hand side, namely

$$|\phi_\alpha\rangle = |\psi_\alpha^+\rangle - \frac{1}{E_\alpha - H_0 + i\epsilon} H_I |\psi_\alpha^+\rangle.$$

Now, let's apply on both sides the operator $(E_\alpha - H + i\epsilon)^{-1} H_I$, where H is the complete Hamiltonian and H_I is the interaction potential. We recall that the asymptotic states $|\psi_\alpha^+\rangle$ are eigenstates of H with eigenvalue E_α, while $|\phi_\alpha\rangle$ are eigenstates of the free Hamiltonian $H_0 \equiv H - H_I$ with the same eigenvalue. Then

$$\begin{aligned}
\frac{1}{E_\alpha - H + i\epsilon} H_I |\phi_\alpha\rangle &= \frac{1}{E_\alpha - H + i\epsilon} \left[1 - H_I \frac{1}{E_\alpha - H_0 + i\epsilon} \right] H_I |\psi_\alpha^+\rangle \\
&= \frac{1}{E_\alpha - H + i\epsilon} \left[E_\alpha - H_0 + i\epsilon - H_I \right] \left[\frac{1}{E_\alpha - H_0 + i\epsilon} \right] H_I |\psi_\alpha^+\rangle \\
&= \frac{1}{E_\alpha - H_0 + i\epsilon} H_I |\psi_\alpha^+\rangle,
\end{aligned}$$

where in the last step we have used the definition $H \equiv H_0 + H_I$. The term we have found in the third step is nothing but the one that appears on the right hand side of the Lippman-Schwinger equation, so that at the end we can write

$$|\psi_\alpha^+\rangle = |\phi_\alpha\rangle + \frac{1}{E_\alpha - H + i\epsilon} H_I |\phi_\alpha\rangle,$$

and, consequently, the T-matrix element is

$$T_{\beta\alpha}^+ \equiv \langle\phi_\beta|H_I|\psi_\alpha^+\rangle = \langle\phi_\beta|H_I|\phi_\alpha\rangle + \langle\phi_\beta|H_I \frac{1}{E_\alpha - H + i\epsilon} H_I |\phi_\alpha\rangle. \tag{1}$$

As an application of this formula, let's consider the case in which $H_I = V_I = V_1(\vec{x}) + V_2(\vec{x})$, and in particular $V_2(\vec{x}) = V_1(\vec{x} + \vec{A}) \equiv e^{i\vec{P}\cdot\vec{A}} V_1(\vec{x}) e^{-i\vec{P}\cdot\vec{A}}$. Suppose $V_1(\vec{x})$ to be significantly different from 0 only in a small region around a given point $\vec{x}_0$, so that the points $\vec{x}$ belonging to that region satisfy $|\vec{A}| \gg |\vec{x} - \vec{x}_0|$, i.e. we can neglect multiple scattering.

In particular the product of the two operators V_1, V_2 can be seen to vanish, indeed:

$$V_1 V_2 = \int dx V_1 V_2 |x\rangle\langle x| = \int dx V_1(x) V_2(x) |x\rangle\langle x| \simeq 0$$

Another useful property is the following: consider the operator products coming from Eq.(1), which can be formally expanded in a geometric series, such as:

$$\begin{aligned}
&\frac{1}{E_\alpha - H + i\epsilon} V_1 = \frac{1}{E_\alpha - H_0 - V_1 - V_2 + i\epsilon} V_1 = \frac{1}{E_\alpha - H_0} \frac{1}{1 - \frac{1}{E - H_0}(V_1 + V_2)} V_1 = \\
&\frac{1}{E_\alpha - H_0} \sum_{n=0}^{\infty} \left[\frac{1}{E - H_0}(V_1 + V_2) \right]^n V_1 \simeq \frac{1}{E_\alpha - H_0} \sum_{n=0}^{\infty} \left[\frac{1}{E - H_0} V_1 \right]^n V_1 = \\
&\frac{1}{E_\alpha - H_0 - V_1 + i\epsilon} V_1.
\end{aligned}$$

In the above calculation we assumed that:

$$V_2 \frac{1}{E_\alpha - H_0} V_1 \simeq 0 \tag{2}$$

even though $\frac{1}{E_\alpha - H_0}$ and V_1, V_2 don't commute and $\frac{1}{E_\alpha - H_0}$ is a non-local operator. However in position-space representation the above term can be written as:

$$\int dxdy V_2(x) \frac{1}{\frac{\nabla^2}{2m} + E}(x-y) V_1(y) \propto \int dxdy V_2(x) \frac{e^{-i\sqrt{2mE}|x-y|}}{|x-y|} V_1(y) \tag{3}$$

As both potential are localized in domains separated by a distance $|\vec{A}|$, the term above is suppressed by a factor of $1/|\vec{A}|$. Terms containing such factor give multiple interactions, such that neglecting it leads to the approximation where the incoming state only interacts with V_1 or V_2.

At the end, what one gets is

$$\begin{aligned} T^+_{\beta\alpha} &= \langle\phi_\beta|H_I|\phi_\alpha\rangle + \langle\phi_\beta|H_I \frac{1}{E_\alpha - H + i\epsilon} H_I|\phi_\alpha\rangle \\ &= \langle\phi_\beta|(V_1+V_2)|\phi_\alpha\rangle + \langle\phi_\beta|V_1 \frac{1}{E_\alpha - H_0 - V_1 + i\epsilon} V_1|\phi_\alpha\rangle + \langle\phi_\beta|V_2 \frac{1}{E_\alpha - H_0 - V_2 + i\epsilon} V_2|\phi_\alpha\rangle \\ &= (T_1^+)_{\beta\alpha} + (T_2^+)_{\beta\alpha}. \end{aligned}$$

Now we can express $(T_2^+)_{\beta\alpha}$ in terms of $(T_1^+)_{\beta\alpha}$:

$$\begin{aligned} &(T_2^+)_{\beta\alpha} = \langle\phi_\beta|e^{i\vec{P}\cdot\vec{A}} V_1 e^{-i\vec{P}\cdot\vec{A}}|\phi_\alpha\rangle + \langle\phi_\beta|e^{i\vec{P}\cdot\vec{A}} V_1 e^{-i\vec{P}\cdot\vec{A}} \sum_{n=0}^{\infty} \left[\frac{1}{E-H_0}\right]^{n+1} e^{i\vec{P}\cdot\vec{A}} V_1^n e^{-i\vec{P}\cdot\vec{A}} e^{i\vec{P}\cdot\vec{A}} V_1 e^{-i\vec{P}\cdot\vec{A}}|\phi_\alpha\rangle \\ &= \left(e^{i\vec{P}\cdot\vec{A}} T_1^+ e^{-i\vec{P}\cdot\vec{A}}\right)_{\beta\alpha}, \end{aligned}$$

where we have used the invariance of the free Hamiltonian under translations.

This reasoning can be generalized to an arbitrary number of hamiltonians, so that, in the approximation of large mutual distances, one simply gets

$$H_I = \sum_{j=1}^{N} e^{i\vec{P}\cdot\vec{A}_j} V_1 e^{-i\vec{P}\cdot\vec{A}_j} \quad \Longrightarrow \quad T = \sum_{j=1}^{N} e^{i\vec{P}\cdot\vec{A}_j} V_1 e^{-i\vec{P}\cdot\vec{A}_j},$$

where $\vec{A}_1 \equiv \vec{0}$.
This way, sending a wave with momentum $\vec{k}_i$ on a bunch of scattering potentials, and calling $\vec{q}$ the difference between the outgoing momentum $\vec{k}_f$ and $\vec{k}_i$, one gets

$$\langle\vec{k}_f|T|\vec{k}_i\rangle \equiv F(\vec{q}) = \sum_{j=1}^{N} \langle\vec{k}_f|e^{i\vec{P}\cdot\vec{A}_j} T_1 e^{-i\vec{P}\cdot\vec{A}_j}|\vec{k}_i\rangle = f(\vec{q}) \sum_{j=1}^{N} e^{i\vec{q}\cdot\vec{A}_j},$$

where $f(\vec{q}) \equiv \langle\vec{k}_f|T_1|\vec{k}_i\rangle$. In the continuum limit (looking from distance $|\vec{L}| \gg |\vec{A}_j|$, $\forall\ j$) one has

$$F(\vec{q}) = f(\vec{q}) \int d^3A\ e^{i\vec{q}\cdot\vec{A}} \rho(\vec{A}),$$

which explains why the scattering amplitude $F(\vec{q})$ in a diffraction experiment is the Fourier transform of the matter distribution $\rho(\vec{A})$ (up to an overall form factor $f(\vec{q})$).

Exercise 2: Scattering from a general potential

Suppose we have some state $|\phi_k\rangle$ which is an eigenstate of a free Hamiltonian H_0. For simplicity let us consider $H_0 = \frac{p^2}{2m}$. Let us assume that at certain finite time t and a finite distance L the states start interacting with

a potential V. The system is now described by the full Hamiltonian $H = H_0 + V$. We also assume that the interaction with the potential is localized in space, so that the system, far away from the interaction point and after enough time, can still be described in terms of eigenstates of H_0.
We want to extract an expression for the asymptotic states for this framework. Recalling the Lippmann-Schwinger equation we have:

$$|\Psi_k^\pm\rangle = |\phi_k\rangle + \frac{1}{E_k - H_0 \pm i\epsilon} H_I |\Psi_k^\pm\rangle,$$

where in this case $H_I = V$ and we are labeling the states with the index k: $E_k = \frac{k^2}{2m}$. We can describe the theory in coordinate space taking the bracket with an eigenstate of position operator $\langle x|$ or in momentum space considering the bracket with an eigenstate of momentum $\langle p|$. We choose the former description. Then:

$$\Psi_k^\pm(x) = \langle x|\Psi_k^\pm\rangle = \langle x|\phi_k\rangle + \langle x|\frac{1}{E_k - H_0 \pm i\epsilon} V|\Psi_k^\pm\rangle.$$

Let us insert a complete set of states $\int d^3x'|x'\rangle\langle x'| = 1$ before the operator V:

$$\Psi_k^\pm(x) = \langle x|\phi_k\rangle + \int d^3x' \underbrace{\langle x|\frac{1}{E_k - H_0 \pm i\epsilon}|x'\rangle}_{\mathcal{G}_{k\pm}(x-x')} \langle x'|V|\Psi_k^\pm\rangle.$$

Let us first compute the Green function $\mathcal{G}_\pm(x - x')$:

$$\mathcal{G}_{k\pm}(x - x') = \int d^3p d^3p' \langle x|p\rangle\langle p|\frac{1}{E_k - H_0 \pm i\epsilon}|p'\rangle\langle p'|x'\rangle,$$

where again we inserted two set of complete eigenstates of momentum. Recalling standard results of QM we have:

$$\langle x|p\rangle = \frac{e^{i\vec{x}\cdot\vec{p}}}{\sqrt{(2\pi)^3}},$$

and therefore:

$$\begin{aligned}
\mathcal{G}_{k\pm}(x - x') &= \int \frac{d^3p}{(2\pi)^3} d^3p' \frac{1}{E_k - \frac{p'^2}{2m} \pm i\epsilon} \langle p|p'\rangle e^{i(\vec{p}\cdot\vec{x} - \vec{p}'\cdot\vec{x}')} = \int \frac{d^3p}{(2\pi)^3} d^3p' \frac{1}{E_k - \frac{p'^2}{2m} \pm i\epsilon} \delta^3(\vec{p} - \vec{p}') e^{i(\vec{p}\cdot\vec{x} - \vec{p}'\cdot\vec{x}')} \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_k - \frac{p^2}{2m} \pm i\epsilon} e^{i\vec{p}\cdot(\vec{x} - \vec{x}')},
\end{aligned}$$

where we have used the fact that the free Hamiltonian applied to an eigenstate of momentum simply gives $H_0|p\rangle = \frac{p^2}{2m}|p\rangle$. Hence:

$$\begin{aligned}
\mathcal{G}_{k\pm}(x - x') &= \int \frac{d^3p}{(2\pi)^3} \frac{2m}{k^2 - p^2 \pm i\epsilon} e^{i\vec{p}\cdot(\vec{x} - \vec{x}')} = \int_0^\infty \frac{p^2\, dp}{(2\pi)^2} \frac{2m}{k^2 - p^2 \pm i\epsilon} \int_{-1}^1 d(\cos\theta) e^{ip|\vec{x} - \vec{x}'|\cos\theta} \\
&= \int_0^\infty \frac{p\, dp}{(2\pi)^2} \frac{2m}{i|\vec{x} - \vec{x}'|} \frac{e^{ip|\vec{x} - \vec{x}'|} - e^{-ip|\vec{x} - \vec{x}'|}}{k^2 - p^2 \pm i\epsilon} \\
&= \frac{im}{4\pi^2|\vec{x} - \vec{x}'|} \int_{-\infty}^\infty p\, dp \frac{e^{ip|\vec{x} - \vec{x}'|} - e^{-ip|\vec{x} - \vec{x}'|}}{p^2 - k^2 \mp i\epsilon},
\end{aligned}$$

where in the last step we have used the symmetry of the integrand under $p \to -p$ to extend the integral from $-\infty$ to $+\infty$ and we have therefore divided by 2. The above integral is composed by two pieces and contains two poles at $p^2 = k^2 \pm i\epsilon$ or $p \simeq \pm(k \pm i\epsilon)$ (note that the two $\pm$'s are unrelated and that the ϵ appearing here is *not* the same as before). According to how we close the contour and the $\pm$ prescription we can enclose or not a pole.

Let us separate the discussions and start with $\mathcal{G}_{k+}(x - x')$ where the poles are at $p = \pm(k + i\epsilon)$: for the first term in the integral we close in the upper plane and therefore we encircle only the pole at $p = k + i\epsilon$; for the second piece we close the contour in the lower half-plane and we enclose only the pole at $p = -k - i\epsilon$. Hence:

$$\begin{aligned}
\mathcal{G}_{k+}(x - x') &= \frac{im}{4\pi^2|\vec{x} - \vec{x}'|}(2\pi i) \left[\left(\frac{p\, e^{ip|\vec{x} - \vec{x}'|}}{p + k} \right)\Bigg|_{p=k} + \left(\frac{p\, e^{-ip|\vec{x} - \vec{x}'|}}{p - k} \right)\Bigg|_{p=-k} \right] \\
&= -\frac{2m}{4\pi|\vec{x} - \vec{x}'|} e^{ik|\vec{x} - \vec{x}'|}.
\end{aligned}$$

Let us consider now $\mathcal{G}_{k-}(x-x')$ where the poles are at $p = \pm(k - i\epsilon)$: for the first term in the integral we close again in the upper half-plane and therefore we encircle only the pole at $p = -k + i\epsilon$; for the second piece we close the contour in the lower half-plane and we enclose only the pole at $p = k - i\epsilon$. Hence:

$$\begin{aligned}\mathcal{G}_{k-}(x-x') &= \frac{im}{4\pi^2|\vec{x}-\vec{x}'|}(2\pi i)\left[\left(\frac{p\,e^{ip|\vec{x}-\vec{x}'|}}{p-k}\right)\Bigg|_{p=-k} + \left(\frac{p\,e^{-ip|\vec{x}-\vec{x}'|}}{p+k}\right)\Bigg|_{p=k}\right] \\ &= -\frac{2m}{4\pi|\vec{x}-\vec{x}'|}e^{-ik|\vec{x}-\vec{x}'|}.\end{aligned}$$

At this point we can come back to our Lippmann-Schwinger equation and plug in the expression for the Green function:

$$\Psi_k^{\pm}(x) = \langle x|\phi_k\rangle - \frac{2m}{4\pi}\int d^3x' \frac{e^{\pm ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\langle x'|V|\Psi_k^{\pm}\rangle.$$

Let us now perform the first approximation: we consider the form of $\Psi_k^{\pm}(x)$ for $x \gg L$, where L is the characteristic scale of the potential V (and $x' \leq L$, otherwise the matrix element $\langle x'|V|\Psi_k^{\pm}\rangle$ is 0). Then we can write:

$$|\vec{x}-\vec{x}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta} = r\left(1 - 2\frac{r'}{r}\cos\theta + \frac{r'^2}{r^2}\right)^{1/2} \simeq r\left(1 - \frac{r'}{r}\cos\theta\right) = r - r'\cos\theta,$$

where $r \equiv |\vec{x}|$ and $r' \equiv |\vec{x}'|$ and $\vec{x}\cdot\vec{x}' = rr'\cos\theta$. Finally

$$\frac{e^{\pm ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \simeq \frac{e^{\pm ikr}}{r}e^{\mp ik\hat{x}\cdot\vec{x}'}$$

Plugging this in the expression for $\Psi_k^{\pm}(x)$ and inserting again a complete set of states we have:

$$\Psi_k^{\pm}(x) = \langle x|\phi_k\rangle - \frac{2m}{4\pi}\frac{e^{\pm ikr}}{r}\int d^3x' d^3x'' \langle x'|V|x''\rangle\langle x''|\Psi_k^{\pm}\rangle e^{\mp ik\hat{x}\cdot\vec{x}'}.$$

If the potential is such that $\langle x'|V|x''\rangle = V(x')\langle x'|x''\rangle = V(x')\delta^3(\vec{x}'-\vec{x}'')$, we have

$$\Psi_k^{\pm}(x) = e^{i\vec{k}\cdot\vec{x}} - \frac{2m}{4\pi}\frac{e^{\pm ikr}}{r}\underbrace{\int d^3x' V(x')\langle x'|\Psi_k^{\pm}\rangle e^{\mp ik\hat{x}\cdot\vec{x}'}}_{f(k,\hat{x})}.$$

The above function completely describes the effect of the potential at large distances from it.

Exercise 3: Differential cross section 2→2 in the centre of mass frame

The cross section for a scattering process $AB \to CD$ is given by

$$d\sigma = \frac{1}{4E_A E_B|\vec{v}_A - \vec{v}_B|}|\mathcal{M}_{AB\to CD}|^2 d\Phi_2,$$

where $\mathcal{M}_{AB\to CD}$ is the matrix element associated to the process and $d\Phi_2$ is the 2-body phase space. In general

$$d\Phi_n = \prod_{i=1}^{n}\frac{d^3p_i}{(2\pi)^3 2E_i}(2\pi)^4\delta^4\left(P_A + P_B - \sum_i P_i\right).$$

In our case $i = C, D$ only. In this exercise P_i represents a 4 momentum while p_i is the spatial momentum. Thanks to the presence of the δ^4 we can easily perform 4 integrals in a straightforward way, without caring about the particular form of the matrix element, that here we leave unexpressed. We recall that this definition of the cross section holds for a reference frame in which the velocities of the incoming particles are collinear (the matrix element and the phase space are Lorentz invariant, while the flux factor depends on the reference frame). Let us take the velocities in the $\hat{z}$ direction. Before performing the integrations we can write the flux factor in a different way:

$$\begin{aligned}\frac{1}{4\sqrt{(P_A\cdot P_B)^2 - m_A^2 m_B^2}} &= \frac{1}{4\sqrt{(E_A E_B - p_A^z p_B^z)^2 - (E_A^2 - p_A^{z2})(E_B^2 - p_B^{z2})}} \\ &= \frac{1}{4\sqrt{(E_B p_A^z - E_A p_B^z)^2}} = \frac{1}{4E_A E_B|v_A^z - v_B^z|},\end{aligned}$$

where we have used the definition of velocity $v \equiv p/E$.
Let's now move to the expression of $d\sigma$. We want to compute it in the center of mass frame, which is defined by requiring the sum of the spatial momenta of the colliding particles to be zero: $\vec{p}_A + \vec{p}_B = 0$.
Let us integrate over $d^3 p_D$. This can be done easily using $\delta^3(\vec{p}_A + \vec{p}_B - \vec{p}_C - \vec{p}_D)$. This Dirac delta enforces the momentum conservation $\vec{p}_D = \vec{p}_A + \vec{p}_B - \vec{p}_C = -\vec{p}_C$ and in addition express the energy E_D as a function of the momentum $\vec{p}_C$ we still have to integrate over:

$$E_D = \sqrt{m_D^2 + \vec{p}_D^{\,2}} = \sqrt{m_D^2 + \vec{p}_C^{\,2}}.$$

Hence we get:

$$d\sigma = \frac{1}{4\sqrt{(P_A \cdot P_B)^2 - m_A^2 m_B^2}} |\mathcal{M}_{AB\to CD}|^2 \frac{d^3 p_C}{(2\pi)^3 2E_C 2E_D} (2\pi)\delta\left(E_A + E_B - E_C - E_D\right).$$

There is still a delta function left that we can use to integrate over another variable. Let us pass in polar coordinates and call $p_C \equiv |\vec{p}_C|$:

$$d\sigma = \frac{1}{4\sqrt{(P_A \cdot P_B)^2 - m_A^2 m_B^2}} |\mathcal{M}_{AB\to CD}|^2 \frac{d\varphi\ d\cos\theta\ p_C^2 dp_C}{(2\pi)^3 2E_C 2E_D} (2\pi)\delta\left(E_A + E_B - E_C - E_D\right).$$

In the expression just deduced, the remaining delta function will eliminate one of the three integrations (which we can choose to be the one over dp_C), so that two degrees of freedom remain. If one makes the further assumption of scalar particles or of unpolarized scattering, the process is invariant under azimuthal rotations, so that $|\mathcal{M}_{AB\to CD}|^2$ won't depend on φ, and at the end we will be left with only one effective variable, which can be chosen to coincide with θ: $\cos\theta = \frac{\vec{p}_A \cdot \vec{p}_C}{p_A p_C}$. But in general, without assumptions, two integration variables are left.

Therefore we keep this variable and we integrate over dp_C. This integral is not trivial since the dependence of the delta function on p_C is complicated (p_C enters in E_C and E_D). Let us use the following change of variables:

$$\frac{d(E_C + E_D)}{dp_C} = \frac{d(\sqrt{m_C^2 + p_C^2} + \sqrt{m_D^2 + p_C^2})}{dp_C} = \frac{p_C}{\sqrt{m_C^2 + p_C^2}} + \frac{p_C}{\sqrt{m_D^2 + p_C^2}} = \frac{p_C}{E_C} + \frac{p_C}{E_D}$$
$$\implies \quad dp_C = \frac{E_C E_D}{p_C (E_C + E_D)} d(E_C + E_D).$$

Note that this change of variables gives the same result that we would deduce by using the property of the Dirac delta

$$\delta(f(x) - A) = \frac{\delta(x - \bar{x})}{\left|\frac{df}{dx}\right|_{x=\bar{x}}},$$

where $\bar{x} \equiv f^{-1}(A)$.
Substituting we get:

$$d\sigma = \frac{1}{4\sqrt{(P_A \cdot P_B)^2 - m_A^2 m_B^2}} |\mathcal{M}_{AB\to CD}|^2 \frac{d\varphi\ d\cos\theta\ p_C}{16\pi^2} \frac{d(E_C + E_D)}{E_C + E_D} \delta\left(E_A + E_B - E_C - E_D\right).$$

Note that at this stage the variable p_C is no longer independent (we won't integrate over it), so it must be expressed in terms of the integration variables. In particular one can invert the relation between $E_C + E_D$ and p_C to get

$$p_C \equiv p_C(E_C + E_D) = \sqrt{\frac{(E_C + E_D)^4 + (m_C^2 - m_D^2)^2 - 2(E_C + E_D)^2 (m_C^2 + m_D^2)}{4(E_C + E_D)^2}}.$$

Now we can perform the integral in $d(E_C + E_D)$, which is trivial: calling $X \equiv (E_C + E_D)$ one has

$$\frac{dX}{X} p_C(X) \delta\left(E_A + E_B - X\right) = \frac{p_C(E_A + E_B)}{E_A + E_B}.$$

Finally, let's introduce the Mandelstam variable $s = (P_A + P_B)^2 = (P_C + P_D)^2$. This quantity represents the total center of mass energy squared: $E_A + E_B = E_C + E_D = \sqrt{s}$.
In the end:

$$d\sigma = \frac{1}{4\sqrt{\left(\frac{s - m_A^2 - m_B^2}{2}\right)^2 - m_A^2 m_B^2}} |\mathcal{M}_{AB\to CD}|^2 \frac{d\varphi\ d\cos\theta}{16\pi^2} \frac{p_C(\sqrt{s})}{\sqrt{s}}$$

Using the expression for p_C in the center of mass (the same as above with the substitution $E_C + E_D \to \sqrt{s}$),

$$p_C(\sqrt{s}) = \sqrt{\frac{s^2 + (m_C^2 - m_D^2)^2 - 2s(m_C^2 + m_D^2)}{4s}},$$

we can obtain simple expressions for the particular cases:

$$m_C = m_D = m \qquad d\sigma = \frac{1}{2\sqrt{(s - m_A^2 - m_B^2)^2 - 4m_A^2 m_B^2}} |\mathcal{M}_{AB\to CD}|^2 \frac{d\Omega}{32\pi^2} \sqrt{1 - \frac{4m^2}{s}},$$

$$m_C = m,\ m_D = 0 \qquad d\sigma = \frac{1}{2\sqrt{(s - m_A^2 - m_B^2)^2 - 4m_A^2 m_B^2}} |\mathcal{M}_{AB\to CD}|^2 \frac{d\Omega}{32\pi^2} \left(1 - \frac{m^2}{s}\right),$$

where $d\Omega$ is the solid angle $d\varphi d\cos\theta$.

Exercise 4: Asymptotic States in Quantum Mechanics

Let us consider a one dimensional quantum system. We want to study the scattering from a potential of the general form illustrated in Figure 1: V is significantly different from zero in a region $x \in [-L\,, L]$, while it rapidly approaches 0 for $|x| > L$. Therefore in the regions far away from the potential we can take $V \equiv 0$ and consider

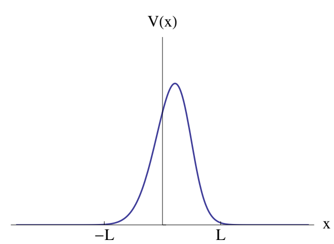

Figure 1: Generic barrier

the theory as free: the solutions with positive energy E_k are complex exponentials:

$$e^{ikx}\,, \qquad e^{-ikx}\,, \qquad E_k = \frac{k^2}{2m}\,,$$

where m is the mass of the particle scattered by the potential. This system represents a simple setup to understand the meaning of *in* and *out* states. Let us first recall their definition in general and then apply it to the this specific case. Given the Hamiltonian H_0, which usually represents the free theory, we call $|\phi_\alpha\rangle$ its eigenvectors:

$$H_0|\phi_\alpha\rangle = E_\alpha|\phi_\alpha\rangle.$$

A general state of the theory will be a wave packet made of a superposition of the states $|\phi_\alpha\rangle$:

$$|\phi\rangle = \int d\alpha\, g(\alpha)|\phi_\alpha\rangle. \tag{4}$$

Let us assume that at a certain time an interaction H_I is switched on. The eigenstates of the complete Hamiltonian $H = H_0 + H_I$ are now modified. Let us assume that they can be labeled with the same index α:

$$(H_0 + H_I)|\psi_\alpha\rangle = E_\alpha|\psi_\alpha\rangle.$$

We can now recall the definition of in- and out- states:

- the *in*-states $|\psi^+\rangle$ tend to $|\phi\rangle$ for $t \to -\infty$;
- the *out*-states $|\psi^-\rangle$ tend to $|\phi\rangle$ for $t \to \infty$.

A more formal definition is given using the Moeller operators. However, as far as this exercise is concerned, the above definition is enough. In the present case the role of H_0 is given by the free Hamiltonian $H_0 = \frac{p^2}{2m}$ while the interaction is $H_I = V(x)$. In a scattering process the states of interest are those representing a particle incoming at time $-\infty$ from the left (or the right) and a particle outgoing at time $+\infty$ to the left (or the right). These states are described by wave packets of the form (4).
We claim that the in-state associated to a incoming particle from the left is given by the following solution:

$$|\psi^+\rangle \equiv \int dk\, g(k)|\psi_k^+\rangle\,, \qquad \langle x|\psi_k^+\rangle = \begin{cases} e^{ikx} + Re^{-ikx}, & \text{for } x \ll -L,\\ Te^{ikx}, & \text{for } x \gg L,\\ \text{not specified} & \text{otherwise,}\end{cases}$$

where R and T are two coefficients. The above state is defined at time $t = 0$. In order to see if the definition of in-state applies we need to evolve it back in time up to $t \to -\infty$. Since the dynamics of $|\psi^+\rangle$ is determined by the full theory we must evolve it using the complete Hamiltonian H. Hence:

$$|\psi^+(t)\rangle = e^{-iHt}|\psi^+\rangle = \int dk\, g(k)e^{-iHt}|\psi_k^+\rangle = \int dk\, g(k)e^{-i\frac{k^2}{2m}t}|\psi_k^+\rangle.$$

Up to now we haven't specified the form of the wave packet $g(k)$. Let us take for simplicity a gaussian distribution centered around a momentum p:

$$g(k) = e^{-\frac{(p-k)^2}{2\sigma^2}}$$

If the gaussian is very narrow the main contribution in the integral on dk will come from the neighborhood of the momentum p and all the rest will be suppressed exponentially. So we can write $k = p + \epsilon$:

$$\begin{aligned}|\psi^+(t)\rangle &= \int dk\, e^{-\frac{(p-k)^2}{2\sigma^2}}\, e^{-i\frac{k^2}{2m}t}|\psi_k^+\rangle \simeq \int dk\, e^{-\frac{(p-k)^2}{2\sigma^2}}\, e^{-i\frac{p^2}{2m}t-i\frac{p\epsilon}{m}t}|\psi_k^+\rangle\,.\\ &\simeq e^{-i\frac{p^2}{2m}t}\int d\epsilon\, e^{-\frac{\epsilon^2}{2\sigma^2}}\, e^{-iv\epsilon t}|\psi_{p+\epsilon}^+\rangle.\end{aligned}$$

where $v \equiv p/m$, and we neglected the term $i\frac{\epsilon^2}{2m}t$ in the exponential.

Let us now consider the above state in the regions $|x| \gg L$, where we make use of the explicit form for $\langle x|\psi_k^+\rangle$. Hence:

$$\langle x|\psi^+(t)\rangle = e^{-i\frac{p^2}{2m}t}\times\begin{cases}\int d\epsilon\, e^{-\frac{\epsilon^2}{2\sigma^2}}\, e^{-iv\epsilon t}\left(e^{i(p+\epsilon)x} + Re^{-i(p+\epsilon)x}\right) = \sqrt{2\pi}\sigma\left(e^{ipx}e^{-\sigma^2\frac{(x-vt)^2}{2}} + Re^{-ipx}e^{-\sigma^2\frac{(x+vt)^2}{2}}\right), & \text{for } x \ll -L,\\ T\int d\epsilon\, e^{-\frac{\epsilon^2}{2\sigma^2}}\, e^{-iv\epsilon t}e^{i(p+\epsilon)x} = T\sqrt{2\pi}\sigma\, e^{ipx}e^{-\sigma^2\frac{(x-vt)^2}{2}}, & \text{for } x \gg L.\end{cases}$$

Finally let us consider the limit $t \to -\infty$: in this limit some terms vanish because they are exponentially suppressed:

$$\langle x|\psi^+(t)\rangle \longrightarrow e^{-i\frac{p^2}{2m}t}\times\begin{cases}\sqrt{2\pi}\sigma\, e^{ipx}e^{-\sigma^2\frac{(x-vt)^2}{2}} & \text{for } x \ll -L,\, t\to-\infty,\\ 0 & \text{for } x \gg L\,, t \to -\infty.\end{cases}$$

The above solution describes as announced an incoming wave packet moving from left to right. Similarly one could find the out-states:

$$|\psi^-\rangle \equiv \int dk\, g(k)|\psi_k^-\rangle\,, \qquad \langle x|\psi_k^-\rangle = \begin{cases} T'e^{+ikx} & \text{for } x \ll -L\\ e^{+ikx} + R'e^{-ikx} & \text{for } x \gg L\\ \text{not specified} & \text{otherwise}\end{cases}$$

Evolving in time as before we get:

$$\langle x|\psi^-(t)\rangle = e^{-i\frac{p^2}{2m}t}\times\begin{cases}T'\int d\epsilon\, e^{-\frac{\epsilon^2}{2\sigma^2}}\, e^{-iv\epsilon t}e^{i(p+\epsilon)x}, & \text{for } x \ll -L,\\ \int d\epsilon\, e^{-\frac{\epsilon^2}{2\sigma^2}}\, e^{-iv\epsilon t}\left(e^{i(p+\epsilon)x} + R'e^{-i(p+\epsilon)x}\right), & \text{for } x \gg L.\end{cases}$$

Again in the limit $t \to \infty$ some integrals vanish and we are left with:

$$\langle x|\psi^-(t)\rangle \longrightarrow \begin{cases} 0 & \text{for } x \ll -L,\\ \sqrt{2\pi}\sigma\, e^{ipx}e^{-\sigma^2\frac{(x-vt)^2}{2}} & \text{for } x \gg L.\end{cases}$$

The above solution represents an outgoing wave packet moving from left to right. One could also find a in-state incoming at time $-\infty$ from the right and out-state escaping towards the left at time $+\infty$. Finally we could compute the matrix element between the in- and out-state, which corresponds to the S-matrix element between the incoming wave packet and the outgoing one:

$$\langle \psi^- | \psi^+ \rangle.$$

Added note: in equation (5) we neglected a piece $-i\frac{\epsilon^2}{2m}t$ in the exponential. One might be worried that this approximation is not justified in the $t \to -\infty$ limit. We can easily restore this factor in the results of the integrals avaluated below simply substituting $1/\sigma^2 \to 1/\sigma^2 + it/m$. Using then

$$Re\left\{ \frac{(x \pm vt)^2}{\frac{1}{\sigma^2} + i\frac{t}{m}} \right\} = \frac{(x \pm vt)^2}{\frac{1}{\sigma^2} + \sigma^2 \frac{t^2}{m^2}} \tag{5}$$

and keeping track only of the real contributions in the exponent, we get

$$\langle x | \psi^+(t) \rangle \sim \begin{cases} \exp\left\{ -\frac{(x-vt)^2/2}{\frac{1}{\sigma^2}+\sigma^2\frac{t^2}{m^2}} \right\} + R\exp\left\{ -\frac{(x+vt)^2/2}{\frac{1}{\sigma^2}+\sigma^2\frac{t^2}{m^2}} \right\}, & \text{for } x \ll -L, \\ T\exp\left\{ -\frac{(x-vt)^2/2}{\frac{1}{\sigma^2}+\sigma^2\frac{t^2}{m^2}} \right\}, & \text{for } x \gg L. \end{cases}$$

As it is well known, in the limit $t \to -\infty$ the wave-packet becomes completely spread. The result (5) still holds in the regime $m \gg |t|\sigma^2$. In practice this is often enough since initial states are prepared at long but finite time before the interaction.