Quantum Field Theory

Set 26: solutions

Exercise 1

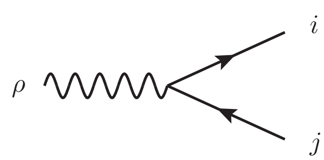
This correction is just a complement to what you can find attached, which is the detailed description of the $e^+e^- \to \mu^+\mu^-$ scattering both in the polarized and in the unpolarized cases, and whose careful reading is highly recommended.
Let's start by recalling the QED Lagrangian as it is usually presented:

$$\mathcal{L}_{QED} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}_e(i\gamma^\rho D_\rho - m_e)\psi_e.$$

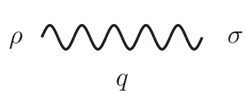
This Lagrangian correctly describes the interaction between photons and electrons, but it is not complete, in the sense that if an experiment can access sufficiently high energies, other leptons, like the muon or the tau, can be produced. So the Lagrangian has to be modified (slightly) to include all these possible interactions

$$\mathcal{L}_{QED} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \sum_l \bar{\psi}_l(i\gamma^\rho D_\rho - m_l)\psi_l,$$

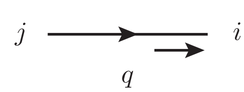
and in this exercise we consider $l = e,\ \mu$. As a matter of notation, we have used $D^\rho = \partial^\rho + ieA^\rho$, where A^ρ is the photon field and e is the coupling constant of QED, namely the positron charge.
From the QED Lagrangian, one can extract the Feynman rules necessary to perform the computation in a diagrammatic way. In this case in the interaction term there is no dependence on the momentum of the particles involved, so it is not even necessary to go to Fourier space in order to extract the factor associated to the vertex: it is just the 'coefficient' of the product of fields that appear in the interaction Lagrangian, times i. Here the interaction Lagrangian is $\mathcal{L}_{int} = -eA_\rho\bar{\psi}_l\gamma^\rho\psi_l$, so the vertex is simply $-ie\gamma^\rho$.
To be complete, we recall all the Feynman rules for QED, including the theory-independent ones (only the vertices are specific of a particular theory: propagators and external legs just depend on the nature (i.e. Poincaré representation) of the particle considered):



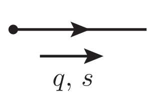
$= -i\,e\,\gamma^\rho_{ij}$



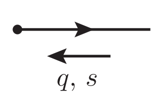
$= \frac{-ig^{\rho\sigma}}{q^2+i\epsilon}$



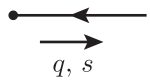
$= \frac{i(q_\rho\gamma^\rho+m)_{ij}}{q^2-m^2+i\epsilon}$



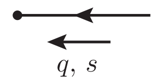
$= \bar{u}^s(q)$



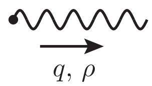
$= \bar{v}^s(q)$



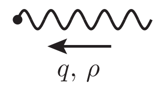
$= v^s(q)$



$= u^s(q)$



$= \epsilon^*_\rho(q)$



$= \epsilon_\rho(q)$

The next step is that of drawing all possible Feynman diagrams allowed by the theory and consistent with the 'topology' that we need, namely an incoming electron-positron pair, and an outgoing muon-antimuon pair. The only diagram for this process (at tree level, i.e. without closed loops) is the one shown at page 131 in Peskin-Shroeder.

At this stage, given the diagrams and the Feynman rules, before performing ANY computation we are already able to guess the high-energy behavior of the cross-section by dimensional analysis. First, we know that the amplitude will be proportional to e^2 since there are two vertices; moreover the cross section has dimensions of E^{-2}; finally, only the $s-$channel is there, and it is always finite, so there are no subtleties linked to the regulation of infrared divergences as we saw in the $\lambda\phi^3$ case. Thus, introducing the fine structure constant $\alpha \equiv \frac{e^2}{4\pi}$, our cross section will be

$$\sigma \sim \frac{\alpha^2}{s},$$

where s is the center of mass energy squared and the numerical coefficient is expected $O(1)$. After the detailed computation we will see that this is indeed the case: this shows how far one can go just by dimensional analysis, which in most situations is enough to assess the order of magnitude of the result.

As last a last supplement to the solution provided by Peskin and Schroeder, we want to show the results in a simple parametrization of momenta. In the center of mass frame one can write

$$\begin{aligned}
p^\mu &= \frac{\sqrt{s}}{2}(1,0,0,1), \\
p'^\mu &= \frac{\sqrt{s}}{2}(1,0,0,-1), \\
k^\mu &= \frac{\sqrt{s}}{2}(1,0,\beta\sin\theta,\beta\cos\theta), \\
k'^\mu &= \frac{\sqrt{s}}{2}(1,0,-\beta\sin\theta,-\beta\cos\theta), \\
q &= p + p',
\end{aligned}$$

where we have approximated the electron as massless and $\beta = \sqrt{1 - 4m_\mu^2/s}$ is the muon velocity in that frame, as can be deduced by imposing $k^2 = k'^2 = m_\mu^2$.

In terms of these variables one has

$$\begin{aligned}
p \cdot p' &= \frac{s}{2}, \\
p \cdot k &= p' \cdot k' = \frac{s}{4}(1 - \beta\cos\theta), \\
p \cdot k' &= p' \cdot k = \frac{s}{4}(1 + \beta\cos\theta), \\
q^2 &= s.
\end{aligned}$$

In terms of these quantities, the amplitude squared becomes

$$\frac{1}{4}\sum_{pol}|\mathcal{M}| \equiv \left|\bar{\mathcal{M}}\right| = \frac{8e^4}{s^2}\left[\frac{s^2}{16}(1+\beta\cos\theta)^2 + \frac{s^2}{16}(1-\beta\cos\theta)^2 + \frac{s}{2}m_\mu^2\right] = e^4\left[2 - \beta^2(1-\cos\theta^2)\right].$$

Given the expression for the phase space for equal mass final particles, and of the flux factor for massless initial particles,

$$d\Phi_2 = \frac{\beta d\cos\theta}{16\pi}, \qquad \mathcal{F} = \frac{1}{2s},$$

at the end the result reads

$$\sigma(s,\beta) = \frac{e^4\beta}{32\pi s}\int_{-1}^{1} dy\ \left(2-\beta^2+\beta^2 y^2\right) \quad = \quad \frac{2\pi\alpha^2}{3s}\beta(3-\beta^2).$$

We can see that the high energy limit of this formula is $\sigma(s,1) = \frac{4\pi\alpha^2}{3s}$, which corresponds to our dimensional analysis guess. Dimensional analysis could determine in few seconds the correct result up to a coefficient $\frac{4\pi}{3} \sim O(1)$.

Exercise 2

Let us consider the following term describing the interaction between a massive vector field Z_μ and a fermion-antifermion pair:

$$\mathcal{L}_{\text{int}} = Z_\mu \bar{l}\gamma^\mu(g_V + g_A\gamma^5)l,$$

where l represents the field associated to a lepton (say the electron). Let us consider parity transformation P. We recall how P acts on fermionic bilinears:

$$\begin{aligned} P: \quad & \bar\psi\gamma^\mu\psi && \longrightarrow && \eta^{\mu\mu}\bar\psi\gamma^\mu\psi && \text{(vector)},\\ & \bar\psi\gamma^\mu\gamma^5\psi && \longrightarrow && -\,\eta^{\mu\mu}\bar\psi\gamma^\mu\gamma^5\psi && \text{(pseudo-vector)}. \end{aligned}$$

Notice that in the above transformation properties the intrinsic parity η_ψ of the fermion ψ doesn't matter since only $|\eta_\psi|^2 = 1$ enters the bilinear. The most general transformation property for the vector field Z_μ is:

$$P:\ Z_\mu \quad\longrightarrow\quad \eta_Z\eta^{\mu\mu}Z_\mu,$$

where we have denoted η_Z the intrinsic parity ($\eta_Z = \pm 1$).
One can easily see that if both g_A and g_V are different from zero there is no choice of η_Z that makes the Lagrangian term $\mathcal{L}_{\text{int}}$ invariant under parity transformations

$$P:\ \mathcal{L}_{\text{int}} \quad\longrightarrow\quad \eta_Z Z_\mu\,\bar{l}\gamma^\mu(g_V - g_A\gamma^5)l.$$

On the contrary, if g_A (g_V) were zero, then the invariance under P would be realized with the choice $\eta_Z = 1$ ($\eta_Z = -1$). In the case in which both coefficients are non vanishing the theory is not parity preserving.
This reasoning has an important consequence: all the observables that measure in some way a parity violation (and are zero in a parity invariant theory) must be proportional to the product $g_V g_A$. They cannot be proportional to just one of the two since setting the other coefficient to zero would make the theory parity invariant while leaving a non vanishing value for the observable, contradicting the fact that the observable must vanish in the parity conserving case.

Let us consider the decay into two leptons, say e^+ and e^-, of a Z particle polarized along the $\hat{z}$ axis. In the center of mass we observe the lepton pair produced back to back. Let us call θ the angle between the momentum of the electron and the $\hat{z}$ axis. One can compute the decay amplitude $\frac{d\Gamma}{d\cos\theta}$ as function of θ. If we perform a parity transformation, this configuration is transformed into the following: the polarization of the Z remains in the same direction, since the spin is a pseudo-vector and doesn't change sign under parity, instead the momenta of the leptons do change sign. This means that the initial process is sent into a similar one in which $\theta' = \pi - \theta$ and therefore:

$$P:\ \frac{d\Gamma}{d\cos\theta}(\theta) \quad\longrightarrow\quad \frac{d\Gamma}{d\cos\theta}(\pi-\theta).$$

If the theory describing this decay were parity preserving then the two processes (with and without the parity transformation) would be the same, and the decay amplitude would be invariant under the above transformation.

When both the coefficients g_V and g_A are non zero the theory is not invariant and it makes sense to compute the difference between the decay width evaluated in the two configuration. In particular one can define the forward-backward asymmetry as:

$$A = \frac{N_+ - N_-}{N_+ + N_-} = \frac{\Gamma_{[0,\pi/2]} - \Gamma_{[\pi/2,\pi]}}{\Gamma_{[0,\pi/2]} + \Gamma_{[\pi/2,\pi]}},$$

where $\Gamma_{[a,b]} = \int_a^b \frac{d\Gamma}{d\theta} d\theta$. The above quantity measures the asymmetry between the number of electrons emitted in the upper half-plane and the number emitted in the lower half-plane. In a parity invariant theory this quantity must be zero, hence we expect it to be proportional to $g_V g_A$.

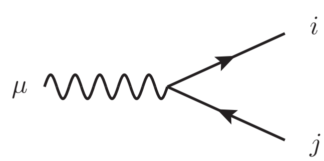
Let us compute the decay width $d\Gamma$; in the center of mass we have

$$d\Gamma = \frac{1}{2M_Z} |\mathcal{M}_{Z\to e^+e^-}|^2 d\Phi_2.$$

As far as the two body phase space is concerned, we can neglect the electron mass, since the two leptons will have an energy $E_+ = E_- = \frac{M_Z}{2} \gg m_e$:

$$d\Gamma = \frac{1}{2M_Z} |\mathcal{M}_{Z\to e^+e^-}|^2 \frac{d\cos\theta}{16\pi}.$$

We now need the expression of the matrix element. The Feynman rule associated to the interaction described by $\mathcal{L}_{\text{int}}$ is:



$$= i[\gamma^\mu(g_V + \gamma^5 g_A)]_{ij}$$

In order to write the expression for the matrix element we start from an external fermionic line and we proceed against the fermion flow direction, associating a factor to the external legs and to the vertex according to the Feynman rules.
Let us come back to our specific example: let us call p_+, p_-, p_Z the momenta of e^+, e^- and Z. Now let us write the matrix element: there is only one fermionic line hence we start from the outgoing electron. We have

- $\bar{u}(p_-)$: is the factor associated to an outgoing particle;
- $i\gamma^\mu(g_V + g_A\gamma^5)$: is the vertex;
- $v(p_+)$ is the factor associated to an outgoing anti-particle;
- ϵ_μ: is the polarization of the external Z.

Thus:

$$i\mathcal{M}_{Z\to e^+e^-} = i\epsilon_\mu\, \bar{u}_-\gamma^\mu(g_V + g_A\gamma^5)v_+,$$

where we have used the short-hand notation u_- (v_+) for $u(p_-)$ ($v(p_+)$). We now perform the square of the matrix element:

$$|\mathcal{M}_{Z\to e^+e^-}|^2 = \epsilon_\mu\epsilon^*_\nu\, [\bar{u}_-\gamma^\mu(g_V + g_A\gamma^5)v_+][\bar{u}_-\gamma^\nu(g_V + g_A\gamma^5)v_+]^\dagger.$$

Note that the amplitude is a $\mathbb{C}-$number, so its complex conjugate corresponds to its hermitian conjugate, that's why we have used $\dagger$ instead of $*$ in the previous expression. The second term in parenthesis can be rewritten as:

$$[\bar{u}_-\gamma^\nu(g_V + g_A\gamma^5)v_+]^\dagger = [u_-^\dagger\gamma^0\gamma^\nu(g_V + g_A\gamma^5)v_+]^\dagger = v_+^\dagger(g_V + g_A\gamma^5)\gamma^{\nu\dagger}\gamma^0 u_-.$$

Recalling that $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$ and $\{\gamma^\nu, \gamma^5\} = 0$ one gets

$$[\bar{u}_-\gamma^\nu(g_V + g_A\gamma^5)v_+]^\dagger = \bar{v}_+\gamma^\nu(g_V + g_A\gamma^5)u_-,$$

and finally the matrix element square reads:

$$|\mathcal{M}_{Z\to e^+e^-}|^2 = \epsilon_\mu\epsilon_\nu^* \, \bar{u}_-\gamma^\mu(g_V + g_A\gamma^5)v_+\bar{v}_+\gamma^\nu(g_V + g_A\gamma^5)u_-.$$

We now perform a trick which allows us to write the matrix element squared as a trace over a set of gamma matrices. This technique is general and holds wherever we deal with fermions in the final or initial state and does not depend on the kind of vertex that we have.
Recalling that u_- and v_+ are four component spinors let us write explicitly their indices:

$$|\mathcal{M}_{Z\to e^+e^-}|^2 = \epsilon_\mu\epsilon_\nu^* \; \bar{u}_{-a}[\gamma^\mu(g_V + g_A\gamma^5)]_{ab}v_{+b}\bar{v}_{+c}[\gamma^\nu(g_V + g_A\gamma^5)]_{cd}u_{-d}.$$

Once we have the indices, we can freely reshuffle terms since we know how to contract them (note also that in this expression there are no annihilation/creation operators since the amplitude is made up of spinors, not fields, so there are no signs appearing from anticommutations of fermions):

$$|\mathcal{M}_{Z\to e^+e^-}|^2 = \epsilon_\mu\epsilon_\nu^* \; \underbrace{u_{-d}\bar{u}_{-a}}_{U_{da}}[\gamma^\mu(g_V + g_A\gamma^5)]_{ab} \underbrace{v_{+b}\bar{v}_{+c}}_{V_{bc}}[\gamma^\nu(g_V + g_A\gamma^5)]_{cd},$$

U and V are 4×4 matrices. One can also rewrite:

$$\begin{aligned}|\mathcal{M}_{Z\to e^+e^-}|^2 &= \epsilon_\mu\epsilon_\nu^* \; [U\gamma^\mu(g_V + g_A\gamma^5)V\gamma^\nu(g_V + g_A\gamma^5)]_{ee} = \epsilon_\mu\epsilon_\nu^* \; \mathrm{Tr}[U\gamma^\mu(g_V + g_A\gamma^5)V\gamma^\nu(g_V + g_A\gamma^5)] \\ &= \epsilon_\mu\epsilon_\nu^* \; \mathrm{Tr}[u_-\bar{u}_-\gamma^\mu(g_V + g_A\gamma^5)v_+\bar{v}_+\gamma^\nu(g_V + g_A\gamma^5)].\end{aligned}$$

Since we are not interested in the polarization of the final electrons (in the measurement of the asymmetry A we only count the number of electrons in a given direction), we can sum over the final polarizations. In doing this we make use of the identities:

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m,$$
$$\sum_s v^s(p)\bar{v}^s(p) = \not{p} - m,$$

where $s = 1, 2$ are the two polarizations. In our case we take the electron as massless, therefore we get:

$$|\bar{\mathcal{M}}_{Z\to e^+e^-}|^2 \equiv \sum_{pol} |\mathcal{M}_{Z\to e^+e^-}|^2 = \epsilon_\mu\epsilon_\nu^* \; \mathrm{Tr}[\not{p}_-\gamma^\mu(g_V + g_A\gamma^5)\not{p}_+\gamma^\nu(g_V + g_A\gamma^5)].$$

Using the relation $\{\gamma^\mu, \gamma^5\} = 0$ we get:

$$\begin{aligned}|\bar{\mathcal{M}}_{Z\to e^+e^-}|^2 &= \epsilon_\mu\epsilon_\nu^* \; \mathrm{Tr}[\not{p}_-\gamma^\mu\not{p}_+\gamma^\nu(g_V + g_A\gamma^5)^2] \\ &= (g_V^2 + g_A^2)\epsilon_\mu\epsilon_\nu^* \; \mathrm{Tr}[\not{p}_-\gamma^\mu\not{p}_+\gamma^\nu] + 2g_Vg_A\epsilon_\mu\epsilon_\nu^* \; \mathrm{Tr}[\not{p}_-\gamma^\mu\not{p}_+\gamma^\nu\gamma^5].\end{aligned}$$

Finally we can make use of the expression for the traces we deduced in Set 25:

$$\begin{aligned}\mathrm{Tr}[\not{p}_-\gamma^\mu\not{p}_+\gamma^\nu] &= 4(p_-^\mu p_+^\nu + p_+^\nu p_-^\mu - \eta^{\mu\nu}p_-\cdot p_+), \\ \mathrm{Tr}[\not{p}_-\gamma^\mu\not{p}_+\gamma^\nu\gamma^5] &= -4i\epsilon^{\alpha\mu\beta\nu}p_{-\alpha}p_{+\beta} = 4i\epsilon^{\mu\nu\alpha\beta}p_{-\alpha}p_{+\beta}.\end{aligned}$$

In order to conclude the computation of the matrix element we need to specify the form of the polarization vector ϵ_μ. We recall that we want to describe a state polarized along the $\hat{z}$ direction. Such a state is of the form:

$$|j_z = 1\rangle = \epsilon_\mu a^{\dagger\mu}|0\rangle,$$

and has to satisfy:

$$J_z|j_z = 1\rangle = |j_z = 1\rangle,$$

where J_z is the generator of the rotation around the $\hat{z}$ direction. We recall that we found its expression in Solution 18: in the rest frame of Z we have:

$$J_z = -i \int d\Omega_{\vec{k}} [\vec{a}^\dagger(\vec{k}) \wedge \vec{a}(\vec{k})]_z = -i \int d\Omega_{\vec{k}} [a_1^\dagger(\vec{k}) a_2(\vec{k}) - a_2^\dagger(\vec{k}) a_1(\vec{k})].$$

One can immediately notice that a state of the form $(a_1^\dagger + i a_2^\dagger)|0\rangle$ is eigenstate of J_z with eigenvalue $+1$, therefore it is a candidate to be the state describing the polarized Z. Hence the polarization vector that gives rise to such a state is :

$$\epsilon_\mu = \frac{1}{\sqrt{2}}(0, 1, i, 0).$$

Using the above expression one can compute the matrix element squared:

$$|\bar{\mathcal{M}}_{Z\to e^+e^-}|^2 = 4(g_V^2 + g_A^2)[(\epsilon \cdot p_-)(\epsilon^* \cdot p_+) + (\epsilon \cdot p_+)(\epsilon^* \cdot p_-) + p_- \cdot p_+] + 8 i g_V g_A \epsilon^{\mu\nu\alpha\beta} p_{-\alpha} p_{+\beta} \epsilon_\mu \epsilon_\nu^*.$$

Let us now express the electron and positron momenta as:

$$p_\pm^\mu = \left(\frac{M_Z}{2}, \pm p^1, \pm p^2, \pm p^3\right), \qquad |\vec{p}| \simeq \frac{M_Z}{2}.$$

Therefore:

$$\begin{aligned}
\epsilon \cdot p_\pm &= \pm \frac{1}{\sqrt{2}}(p^1 + i p^2), \\
\epsilon^* \cdot p_\pm &= \pm \frac{1}{\sqrt{2}}(p^1 - i p^2), \\
p_- \cdot p_+ &= \frac{1}{2}(p_Z^2 - p_-^2 - p_+^2) \simeq \frac{M_Z^2}{2}, \\
\epsilon^{\mu\nu\alpha\beta} p_{-\alpha} p_{+\beta} \epsilon_\mu \epsilon_\nu^* &= -i \epsilon^{12\alpha\beta} p_{-\alpha} p_{+\beta} = i M_Z p^3 \simeq \frac{i}{2} M_Z^2 \cos\theta.
\end{aligned}$$

Finally, writing $p_1^2 + p_2^2 = \frac{M_Z^2}{4}\sin^2\theta$, we have

$$\begin{aligned}
|\bar{\mathcal{M}}_{Z\to e^+e^-}|^2 &= M_Z^2 (g_V^2 + g_A^2)[2 - \sin^2\theta] - 4 M_Z^2 g_V g_A \cos\theta \\
&= M_Z^2 (g_V^2 + g_A^2)\left[1 + \cos^2\theta - 4\frac{g_V g_A}{g_V^2 + g_A^2}\cos\theta\right].
\end{aligned}$$

In the end

$$d\Gamma = \frac{M_Z}{32\pi}(g_V^2 + g_A^2)\left[1 + \cos^2\theta - 4\frac{g_V g_A}{g_V^2 + g_A^2}\cos\theta\right] d\cos\theta.$$

Now we are ready to compute the asymmetry:

$$\begin{aligned}
\Gamma_{[0,\pi/2]} + \Gamma_{[\pi/2,\pi]} = \Gamma &= \frac{M_Z}{32\pi}(g_V^2 + g_A^2)\int_{-1}^{1}(1 + y^2)dy = \frac{M_Z}{32\pi}(g_V^2 + g_A^2) \times \frac{8}{3}, \\
\Gamma_{[0,\pi/2]} - \Gamma_{[\pi/2,\pi]} &= \frac{M_Z}{32\pi}(2 g_V g_A)(-4)\int_0^1 y\, dy = -4\frac{M_Z}{32\pi} g_V g_A.
\end{aligned}$$

In the end the up-down asymmetry is given by simply:

$$A = -\frac{3}{4}\frac{2 g_V g_A}{g_V^2 + g_A^2}$$

and as expected it is proportional to both couplings g_V and g_A.

localized potential is

$$d\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\,|\mathcal{M}(p_i \to p_f)|^2\,(2\pi)\delta(E_f - E_i),$$

where v_i is the particle's initial velocity. This formula is a natural modification of (4.79). Integrate over $|p_f|$ to find a simple expression for $d\sigma/d\Omega$.

(c) Specialize to the case of electron scattering from a Coulomb potential ($A^0 = Ze/4\pi r$). Working in the nonrelativistic limit, derive the Rutherford formula,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4m^2v^4\sin^4(\theta/2)}.$$

(With a few calculational tricks from Section 5.1, you will have no difficulty evaluating the general cross section in the relativistic case; see Problem 5.1.)

Chapter 5

Elementary Processes of Quantum Electrodynamics

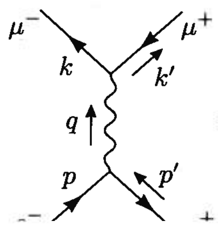
Finally, after three long chapters of formalism, we are ready to perform some real relativistic calculations, to begin working out the predictions of Quantum Electrodynamics. First we will return to the process considered in Chapter 1, the annihilation of an electron-positron pair into a pair of heavier fermions. We will study this paradigm process in extreme detail in the next three sections, then do a few more simple QED calculations in Sections 5.4 and 5.5. The problems at the end of the chapter treat several additional QED processes. More complete surveys of QED can be found in the books of Jauch and Rohrlich (1976) and of Berestetskii, Lifshitz, and Pitaevskii (1982).

5.1 $e^+e^- \to \mu^+\mu^-$: Introduction

The reaction $e^+e^- \to \mu^+\mu^-$ is the simplest of all QED processes, but also one of the most important in high-energy physics. It is fundamental to the understanding of all reactions in e^+e^- colliders, and is in fact used to calibrate such machines. The related process $e^+e^- \to q\bar{q}$ (a quark-antiquark pair) is extraordinarily useful in determining the properties of elementary particles.

In this section we will compute the *unpolarized* cross section for $e^+e^- \to \mu^+\mu^-$, to lowest order. In Chapter 1 we used elementary arguments to guess the answer (Eq. (1.8)) in the limit where all the fermions are massless. We now relax that restriction and retain the muon mass in the calculation. Retaining the electron mass as well would be easy but pointless, since the ratio $m_e/m_\mu \approx 1/200$ is much smaller than the fractional error introduced by neglecting higher-order terms in the perturbation series.

Using the Feynman rules from Section 4.8, we can at once draw the diagram and write down the amplitude for our process:



$$= \bar{v}^{s'}(p')(-ie\gamma^\mu)u^s(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}^r(k)(-ie\gamma^\nu)v^{r'}(k').$$

Rearranging this slightly and leaving the spin superscripts implicit, we have

$$i\mathcal{M}\big(e^-(p)e^+(p')\to\mu^-(k)\mu^+(k')\big)=\frac{ie^2}{q^2}\Big(\bar{v}(p')\gamma^\mu u(p)\Big)\Big(\bar{u}(k)\gamma_\mu v(k')\Big).\qquad(5.1)$$

This answer for the amplitude $\mathcal{M}$ is simple, but not yet very illuminating.

To compute the differential cross section, we need an expression for $|\mathcal{M}|^2$, so we must find the complex conjugate of $\mathcal{M}$. A bi-spinor product such as $\bar{v}\gamma^\mu u$ can be complex-conjugated as follows:

$$(\bar{v}\gamma^\mu u)^* = u^\dagger(\gamma^\mu)^\dagger(\gamma^0)^\dagger v = u^\dagger(\gamma^\mu)^\dagger\gamma^0 v = u^\dagger\gamma^0\gamma^\mu v = \bar{u}\gamma^\mu v.$$

(This is another advantage of the 'bar' notation.) Thus the squared matrix element is

$$|\mathcal{M}|^2=\frac{e^4}{q^4}\Big(\bar{v}(p')\gamma^\mu u(p)\bar{u}(p)\gamma^\nu v(p')\Big)\Big(\bar{u}(k)\gamma_\mu v(k')\bar{v}(k')\gamma_\nu u(k)\Big).\qquad(5.2)$$

At this point we are still free to specify any particular spinors $u^s(p)$, $\bar{v}^{s'}(p')$, and so on, corresponding to any desired spin states of the fermions. In actual experiments, however, it is difficult (though not impossible) to retain control over spin states; one would have to prepare the initial state from polarized materials and/or analyze the final state using spin-dependent multiple scattering. In most experiments the electron and positron beams are unpolarized, so the measured cross section is an *average* over the electron and positron spins s and s'. Muon detectors are normally blind to polarization, so the measured cross section is a *sum* over the muon spins r and r'.

The expression for $|\mathcal{M}|^2$ simplifies considerably when we throw away the spin information. We want to compute

$$\frac{1}{2}\sum_s\frac{1}{2}\sum_{s'}\sum_r\sum_{r'}\big|\mathcal{M}(s,s'\to r,r')\big|^2.$$

The spin sums can be performed using the completeness relations from Section 3.3:

$$\sum_s u^s(p)\bar{u}^s(p)=\not{p}+m;\qquad \sum_s v^s(p)\bar{v}^s(p)=\not{p}-m.\qquad(5.3)$$

Working with the first half of (5.2), and writing in spinor indices so we can freely move the v next to the $\bar{v}$, we have

$$\begin{aligned}\sum_{s,s'}\bar{v}^{s'}_a(p')\gamma^\mu_{ab}u^s_b(p)\bar{u}^s_c(p)\gamma^\nu_{cd}v^{s'}_d(p') &= (\not{p}'-m)_{da}\gamma^\mu_{ab}(\not{p}+m)_{bc}\gamma^\nu_{cd}\\ &=\text{trace}\big[(\not{p}'-m)\gamma^\mu(\not{p}+m)\gamma^\nu\big].\end{aligned}$$

Evaluating the second half of (5.2) in the same way, we arrive at the desired simplification:

$$\frac{1}{4}\sum|\mathcal{M}|^2=\frac{e^4}{4q^4}\,\mathrm{tr}\big[(\not{p}'-m_e)\gamma^\mu(\not{p}+m_e)\gamma^\nu\big]\,\mathrm{tr}\big[(\not{k}+m_\mu)\gamma_\mu(\not{k}'-m_\mu)\gamma_\nu\big].$$

The spinors u and v have disappeared, leaving us with a much cleaner expression in terms of γ matrices. This trick is very general: Any QED amplitude involving external fermions, when squared and summed or averaged over spins, can be converted in this way to traces of products of Dirac matrices.

Trace Technology

This last step would hardly be an improvement if the traces had to be laboriously computed by brute force. But Feynman found that they could be worked out easily by appealing to the algebraic properties of the γ matrices. Since the evaluation of such traces occurs so often in QED calculations, it is worthwhile to pause and attack the problem systematically, once and for all.

We would like to evaluate traces of products of n gamma matrices, where $n=0,1,2,\ldots$. (For the present problem we need $n=2,3,4$.) The $n=0$ case is fairly easy: $\mathrm{tr}\,\mathbf{1}=4$. The trace of one γ matrix is also easy. From the explicit form of the matrices in the chiral representation, we have

$$\mathrm{tr}\,\gamma^\mu=\mathrm{tr}\begin{pmatrix}0 & \sigma^\mu\\ \bar{\sigma}^\mu & 0\end{pmatrix}=0.$$

It is useful to prove this result in a more abstract way, which generalizes to an arbitrary odd number of γ matrices:

$$\begin{aligned}\mathrm{tr}\,\gamma^\mu &= \mathrm{tr}\,\gamma^5\gamma^5\gamma^\mu &&\text{since } (\gamma^5)^2=1\\ &=-\mathrm{tr}\,\gamma^5\gamma^\mu\gamma^5 &&\text{since } \{\gamma^\mu,\gamma^5\}=0\\ &=-\mathrm{tr}\,\gamma^5\gamma^5\gamma^\mu &&\text{using cyclic property of trace}\\ &=-\mathrm{tr}\,\gamma^\mu.\end{aligned}$$

Since the trace of γ^μ is equal to minus itself, it must vanish. For n γ-matrices we would get n minus signs in the second step (as we move the second γ^5 all the way to the right), so the trace must vanish if n is odd.

To evaluate the trace of two γ matrices, we again use the anticommutation properties and the cyclic property of the trace:

$$\begin{aligned}\mathrm{tr}\,\gamma^\mu\gamma^\nu &= \mathrm{tr}\big(2g^{\mu\nu}\cdot\mathbf{1}-\gamma^\nu\gamma^\mu\big) &&\text{(anticommutation)}\\ &=8g^{\mu\nu}-\mathrm{tr}\,\gamma^\mu\gamma^\nu &&\text{(cyclicity)}\end{aligned}$$

Thus $\mathrm{tr}\,\gamma^\mu\gamma^\nu=4g^{\mu\nu}$. The trace of any even number of γ matrices can be evaluated in the same way: Anticommute the first γ matrix all the way to the right, then cycle it back to the left. Thus for the trace of four γ matrices, we have

$$\mathrm{tr}\big(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\big)=\mathrm{tr}\big(2g^{\mu\nu}\gamma^\rho\gamma^\sigma-\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\big)$$

Using the cyclic property on the last term and bringing it to the left-hand side, we find

$$\begin{aligned}\operatorname{tr}\big(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\big) &= g^{\mu\nu}\operatorname{tr}\gamma^\rho\gamma^\sigma - g^{\mu\rho}\operatorname{tr}\gamma^\nu\gamma^\sigma + g^{\mu\sigma}\operatorname{tr}\gamma^\nu\gamma^\rho \\ &= 4\big(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\big).\end{aligned}$$

In this manner one can always reduce a trace of n γ-matrices to a sum of traces of $(n-2)$ γ-matrices. The case $n=6$ is easy to work out, but has fifteen terms (the number of ways of grouping the six indices in pairs to make terms of the form $g^{\mu\nu}g^{\rho\sigma}g^{\alpha\beta}$). Fortunately, we will not need it in this book. (If you ever do need to evaluate such complicated traces, it may be easier to learn to use one of the several computer programs that can perform symbolic manipulations on Dirac matrices.)

Starting in Section 5.2, we will often need to evaluate traces involving γ^5. Since $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, the trace of γ^5 times any odd number of other γ matrices is zero. It is also easy to show that the trace of γ^5 itself is zero:

$$\operatorname{tr}\gamma^5 = \operatorname{tr}\big(\gamma^0\gamma^0\gamma^5\big) = -\operatorname{tr}\big(\gamma^0\gamma^5\gamma^0\big) = -\operatorname{tr}\big(\gamma^0\gamma^0\gamma^5\big) = -\operatorname{tr}\gamma^5.$$

The same trick works for $\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^5)$, if we insert two factors of γ^α for some α different from both μ and ν. The first nonvanishing trace involving γ^5 contains four other γ matrices. In this case the trick still works unless every γ matrix appears, so $\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = 0$ unless $(\mu\nu\rho\sigma)$ is some permutation of (0123). From the anticommutation rules it also follows that interchanging any two of the indices simply changes the sign of the trace, so $\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5)$ must be proportional to $\epsilon^{\mu\nu\rho\sigma}$. The overall constant turns out to be $-4i$, as you can easily check by plugging in $(\mu\nu\rho\sigma) = (0123)$.

Here is a summary of the trace theorems, for convenient reference:

$$\begin{aligned}\operatorname{tr}(\mathbf{1}) &= 4 \\ \operatorname{tr}(\text{any odd \# of }\gamma\text{'s}) &= 0 \\ \operatorname{tr}(\gamma^\mu\gamma^\nu) &= 4g^{\mu\nu} \\ \operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \\ \operatorname{tr}(\gamma^5) &= 0 \\ \operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^5) &= 0 \\ \operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) &= -4i\epsilon^{\mu\nu\rho\sigma}\end{aligned} \tag{5.5}$$

Expressions resulting from use of the last formula can be simplified by means of the identities

$$\begin{aligned}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} &= -24 \\ \epsilon^{\alpha\beta\gamma\mu}\epsilon_{\alpha\beta\gamma\nu} &= -6\delta^\mu{}_\nu \\ \epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} &= -2\big(\delta^\mu{}_\rho\delta^\nu{}_\sigma - \delta^\mu{}_\sigma\delta^\nu{}_\rho\big)\end{aligned} \tag{5.6}$$

Another useful identity allows one to reverse the order of all the γ matrices inside a trace:

$$\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\cdots) = \operatorname{tr}(\cdots\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu). \tag{5.7}$$

To prove this relation, consider the matrix $C \equiv \gamma^0\gamma^2$ (essentially the charge-conjugation operator). This matrix satisfies $C^2 = 1$ and $C\gamma^\mu C = -(\gamma^\mu)^T$. Thus if there are n γ-matrices inside the trace,

$$\begin{aligned}\operatorname{tr}(\gamma^\mu\gamma^\nu\cdots) &= \operatorname{tr}(C\gamma^\mu C\,C\gamma^\nu C\cdots) \\ &= (-1)^n\operatorname{tr}\big[(\gamma^\mu)^T(\gamma^\nu)^T\cdots\big] \\ &= \operatorname{tr}(\cdots\gamma^\nu\gamma^\mu),\end{aligned}$$

since the trace vanishes unless n is even. It is easy to show that the reversal identity (5.7) is also valid when the trace contains one or more factors of γ^5.

When two γ matrices inside a trace are dotted together, it is easiest to eliminate them before evaluating the trace. For example,

$$\gamma^\mu\gamma_\mu = g_{\mu\nu}\gamma^\mu\gamma^\nu = \tfrac{1}{2}g_{\mu\nu}\{\gamma^\mu,\gamma^\nu\} = g_{\mu\nu}g^{\mu\nu} = 4. \tag{5.8}$$

The following *contraction identities*, all easy to prove using the anticommutation relations, can be used when other γ matrices lie in between:

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma_\mu &= -2\gamma^\nu \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4g^{\nu\rho} \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu\end{aligned} \tag{5.9}$$

Note the reversal of order in the last identity.

All of the γ matrix identities proved in this section are collected for reference in the Appendix.

Unpolarized Cross Section

We now return to the evaluation of the squared matrix element, Eq. (5.4). The electron trace is

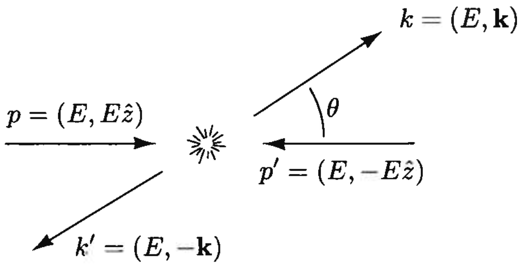
$$\operatorname{tr}\big[(\not p' - m_e)\gamma^\mu(\not p + m_e)\gamma^\nu\big] = 4\big[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p\cdot p' + m_e^2)\big].$$

The terms with only one factor of m vanish, since they contain an odd number of γ matrices. Similarly, the muon trace is

$$\operatorname{tr}\big[(\not k + m_\mu)\gamma_\mu(\not k' - m_\mu)\gamma_\nu\big] = 4\big[k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}(k\cdot k' + m_\mu^2)\big].$$

From now on we will set $m_e = 0$, as discussed at the beginning of this section. Dotting these expressions together and collecting terms, we get the simple result

To obtain a more explicit formula we must specialize to a particular frame of reference and express the vectors p, p', k, k', and q in terms of the basic kinematic variables—energies and angles—in that frame. In practice, the choice of frame will be dictated by the experimental conditions. In this book, we will usually make the simplest choice of evaluating cross sections in the center-of-mass frame. For this choice, the initial and final 4-momenta for $e^+e^- \to \mu^+\mu^-$ can be written as follows:



$$|\mathbf{k}| = \sqrt{E^2 - m_\mu^2}$$

$$\mathbf{k}\cdot\hat{z} = |\mathbf{k}|\cos\theta$$

To compute the squared matrix element we need

$$q^2 = (p+p')^2 = 4E^2; \qquad p\cdot p' = 2E^2;$$
$$p\cdot k = p'\cdot k' = E^2 - E|\mathbf{k}|\cos\theta; \qquad p\cdot k' = p'\cdot k = E^2 + E|\mathbf{k}|\cos\theta.$$

We can now rewrite Eq. (5.10) in terms of E and θ:

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{8e^4}{16E^4}\Big[E^2(E-|\mathbf{k}|\cos\theta)^2 + E^2(E+|\mathbf{k}|\cos\theta)^2 + 2m_\mu^2E^2\Big]\\ &= e^4\left[\left(1+\frac{m_\mu^2}{E^2}\right) + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right].\end{aligned} \tag{5.11}$$

All that remains is to plug this expression into the cross-section formula derived in Section 4.5. Since there are only two particles in the final state and we are working in the center-of-mass frame, we can use the simplified formula (4.84). For our problem $|v_\mathcal{A} - v_\mathcal{B}| = 2$ and $E_\mathcal{A} = E_\mathcal{B} = E_{\text{cm}}/2$, so we have

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{1}{2E_{\text{cm}}^2}\,\frac{|\mathbf{k}|}{16\pi^2E_{\text{cm}}}\cdot\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\\ &= \frac{\alpha^2}{4E_{\text{cm}}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left[\left(1+\frac{m_\mu^2}{E^2}\right) + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right].\end{aligned} \tag{5.12}$$

Integrating over $d\Omega$, we find the total cross section:

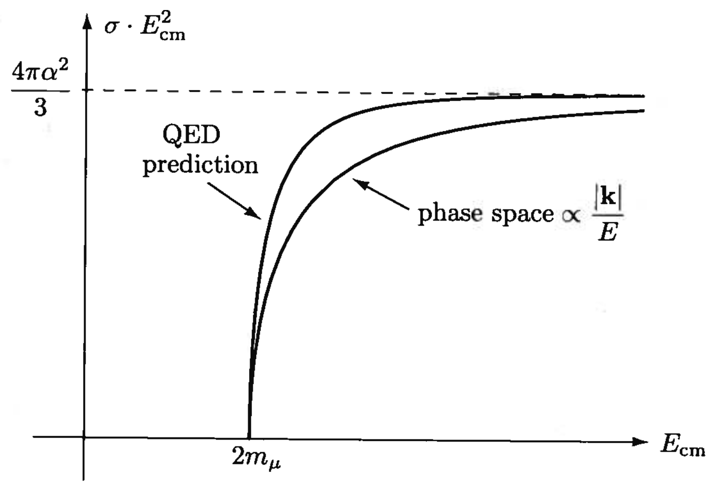


Figure 5.1. Energy dependence of the total cross section for $e^+e^- \to \mu^+\mu^-$, compared to "phase space" energy dependence.

In the high-energy limit where $E \gg m_\mu$, these formulae reduce to those given in Chapter 1:

$$\begin{aligned}\frac{d\sigma}{d\Omega} &\xrightarrow[E\gg m_\mu]{} \frac{\alpha^2}{4E_{\text{cm}}^2}(1+\cos^2\theta);\\ \sigma_{\text{total}} &\xrightarrow[E\gg m_\mu]{} \frac{4\pi\alpha^2}{3E_{\text{cm}}^2}\left(1-\frac{3}{8}\left(\frac{m_\mu}{E}\right)^4 - \cdots\right).\end{aligned} \tag{5.14}$$

Note that these expressions have the correct dimensions of cross sections. In the high-energy limit, E_{cm} is the only dimensionful quantity in the problem, so dimensional analysis dictates that $\sigma_{\text{total}} \propto E_{\text{cm}}^{-2}$. Since we knew from the beginning that $\sigma_{\text{total}} \propto \alpha^2$, we only had to work to get the factor of $4\pi/3$.

The energy dependence of the total cross-section formula (5.13) near threshold is shown in Fig. 5.1. Of course the cross section is zero for $E_{\text{cm}} < 2m_\mu$. It is interesting to compare the shape of the actual curve to the shape one would obtain if $|\mathcal{M}|^2$ did not depend on energy, that is, if all the energy dependence came from the phase-space factor $|\mathbf{k}|/E$. To test Quantum Electrodynamics, an experiment must be able to resolve deviations from the naive phase-space prediction. Experimental results from pair production of both μ and τ leptons confirm that these particles behave as QED predicts. Figure 5.2 compares formula (5.13) to experimental measurements of the $\tau^+\tau^-$ threshold.

Before discussing our result further, let us pause to summarize how we obtained it. The method extends in a straightforward way to the calculation

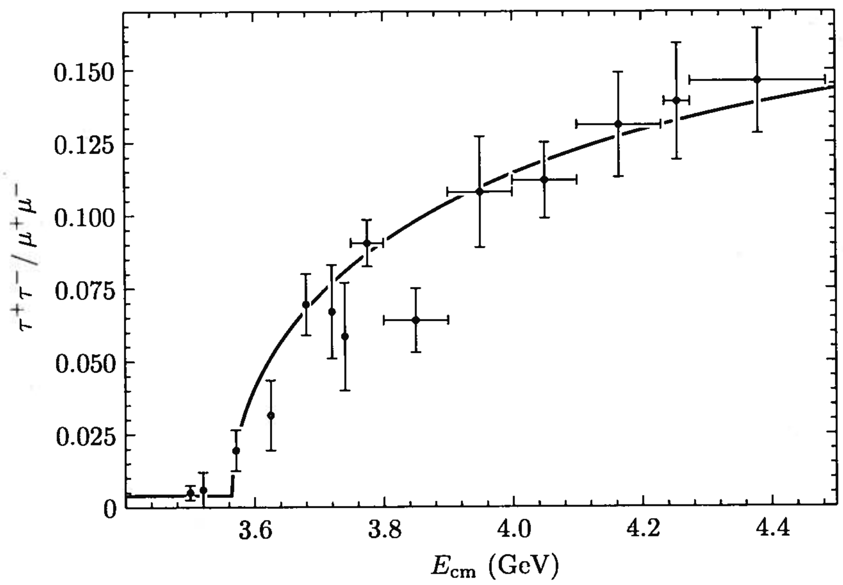


Figure 5.2. The ratio $\sigma(e^+e^- \to \tau^+\tau^-)/\sigma(e^+e^- \to \mu^+\mu^-)$ of measured cross sections near the threshold for $\tau^+\tau^-$ pair-production, as measured by the DELCO collaboration, W. Bacino, et. al., *Phys. Rev. Lett.* **41**, 13 (1978). Only a fraction of τ decays are included, hence the small overall scale. The curve shows a fit to the theoretical formula (5.13), with a small energy-independent background added. The fit yields $m_\tau = 1782^{+2}_{-7}$ MeV.

1. Draw the diagram(s) for the desired process.
2. Use the Feynman rules to write down the amplitude $\mathcal{M}$.
3. Square the amplitude and average or sum over spins, using the completeness relations (5.3). (For processes involving photons in the final state there is an analogous completeness relation, derived in Section 5.5.)
4. Evaluate traces using the trace theorems (5.5); collect terms and simplify the answer as much as possible.
5. Specialize to a particular frame of reference, and draw a picture of the kinematic variables in that frame. Express all 4-momentum vectors in terms of a suitably chosen set of variables such as E and θ.
6. Plug the resulting expression for $|\mathcal{M}|^2$ into the cross-section formula (4.79), and integrate over phase-space variables that are not measured to obtain a differential cross section in the desired form. (In our case these integrations were over the constrained momenta $\mathbf{k}'$ and $|\mathbf{k}|$, and were performed in the derivation of Eq. (4.84).)

While other calculations (especially those involving loop diagrams) often re-

Production of Quark-Antiquark Pairs

The asymptotic energy dependence of the $e^+e^- \to \mu^+\mu^-$ cross-section formula sets the scale for all e^+e^- annihilation cross sections. A particularly important example is the cross section for

$$e^+e^- \to \text{hadrons},$$

that is, the total cross section for production of any number of strongly interacting particles.

In our current understanding of the strong interactions, given by the theory called Quantum Chromodynamics (QCD), all hadrons are composed of Dirac fermions called *quarks*. Quarks appear in a variety of types, called *flavors*, each with its own mass and electric charge. A quark also carries an additional quantum number, *color*, which takes one of three values. Color serves as the "charge" of QCD, as we will discuss in Chapter 17.

According to QCD, the simplest e^+e^- process that ends in hadrons is

$$e^+e^- \to q\bar{q},$$

the annihilation of an electron and a positron, through a virtual photon, into a quark-antiquark pair. After they are created, the quarks interact with one another through their strong forces, producing more quark pairs. Eventually the quarks and antiquarks combine to form some number of mesons and baryons.

To adapt our results for muon production to handle the case of quarks, we must make three modifications:

1. Replace the muon charge e with the quark charge $Q|e|$.
2. Count each quark three times, one for each color.
3. Include the effects of the strong interactions of the produced quark and antiquark.

The first two changes are easy to make. For the first, it is simply necessary to know the masses and charges of each flavor of quark. For u, c, and t quarks we have $Q = 2/3$, while for d, s, and b quarks we have $Q = -1/3$. The cross-section formulae are proportional to the square of the charge of the final-state particle, so we can simply insert a factor of Q^2 into any of these formulae to obtain the cross section for production of any particular variety of quark. Counting colors is necessary because experiments measure only the total cross section for production of all three colors. (The hadrons that are actually detected are colorless.) In any case, this counting is easy: Just multiply the answer by 3.

If you know a little about the strong interaction, however, you might think this is all a big joke. Surely the third modification is extremely difficult to make, and will drastically alter the predictions of QED. The amazing truth is that in the high-energy limit, the effect of the strong interaction on the up the final-state quarks into bunches of hadrons. This simplification is due to a phenomenon called *asymptotic freedom*; it played a crucial role in the identification of Quantum Chromodynamics as the correct theory of the strong force.

Thus in the high-energy limit, we expect the cross section for the reaction $e^+e^- \to q\bar{q}$ to approach $3 \cdot Q^2 \cdot 4\pi\alpha^2/3E_{\rm cm}^2$. It is conventional to define

$$1 \text{ unit of R} \equiv \frac{4\pi\alpha^2}{3E_{\rm cm}^2} = \frac{86.8\,\text{nbarns}}{(E_{\rm cm}\text{ in GeV})^2}. \tag{5.15}$$

The value of a cross section in units of R is therefore its ratio to the asymptotic value of the $e^+e^- \to \mu^+\mu^-$ cross section predicted by Eq. (5.14). Experimentally, the easiest quantity to measure is the total rate for production of all hadrons. Asymptotically, we expect

$$\sigma(e^+e^- \to \text{hadrons}) \underset{E_{\rm cm}\to\infty}{\longrightarrow} 3 \cdot \Big(\sum_i Q_i^2\Big)\ \text{R}, \tag{5.16}$$

where the sum runs over all quarks whose masses are smaller than $E_{\rm cm}/2$. When $E_{\rm cm}/2$ is in the vicinity of one of the quark masses, the strong interactions cause large deviations from this formula. The most dramatic such effect is the appearance of *bound states* just below $E_{\rm cm} = 2m_q$, manifested as very sharp spikes in the cross section.

Experimental measurements of the cross section for e^+e^- annihilation to hadrons between 2.5 and 40 GeV are shown in Fig. 5.3. The data shows three distinct regions: a low-energy region in which u, d, and s quark pairs are produced; a region above the threshold for production of c quark pairs; and a region also above the threshold for b quark pairs. The prediction (5.16) is shown as a set of solid lines; it agrees quite well with the data in each region, as long as the energy is well away from the thresholds where the high-energy approximation breaks down. The dotted curves show an improved theoretical prediction, including higher-order corrections from QCD, which we will discuss in Section 17.2. This explanation of the e^+e^- annihilation cross section is a remarkable success of QCD. In particular, experimental verification of the factor of 3 in (5.16) is one piece of evidence for the existence of color.

The angular dependence of the differential cross section is also observed experimentally.* At high energy the hadrons appear in *jets*, clusters of several hadrons all moving in approximately the same direction. In most cases there are two jets, with back-to-back momenta, and these indeed have the angular dependence $(1 + \cos^2\theta)$.

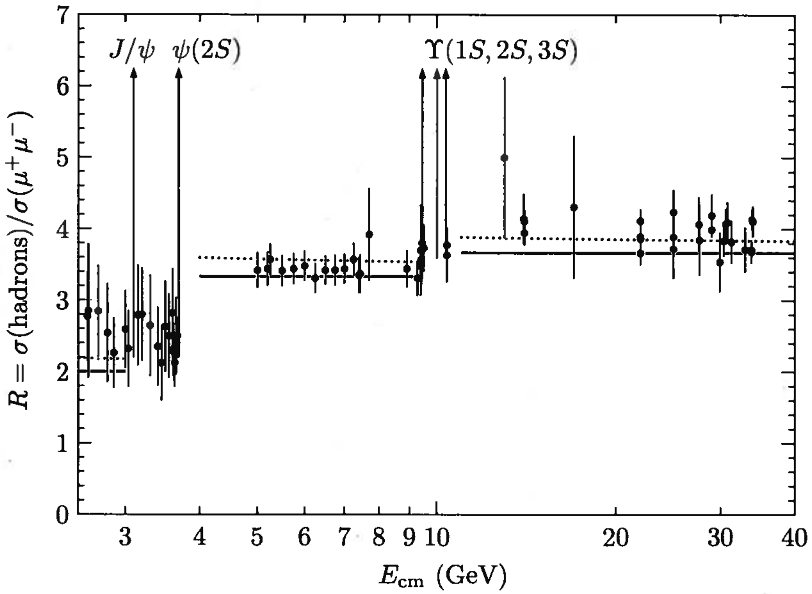


Figure 5.3. Experimental measurements of the total cross section for the reaction $e^+e^- \to$ hadrons, from the data compilation of M. Swartz, *Phys. Rev.* **D53**, 5268 (1996). Complete references to the various experiments are given there. The measurements are compared to theoretical predictions from Quantum Chromodynamics, as explained in the text. The solid line is the simple prediction (5.16).

5.2 $e^+e^- \to \mu^+\mu^-$: Helicity Structure

The unpolarized cross section for a reaction is generally easy to calculate (and to measure) but hard to understand. Where does the $(1 + \cos^2\theta)$ angular dependence come from? We can answer this question by computing the $e^+e^- \to \mu^+\mu^-$ cross section for each set of spin orientations separately.

First we must choose a basis of polarization states. To get a simple answer in the high-energy limit, the best choice is to quantize each spin along the direction of the particle's motion, that is, to use states of definite helicity. Recall that in the massless limit, the left- and right-handed helicity states of a Dirac particle live in different representations of the Lorentz group. We might therefore expect them to behave independently, and in fact they do.

In this section we will compute the polarized $e^+e^- \to \mu^+\mu^-$ cross sections, using the helicity basis, in two different ways: first, by using trace technology but with the addition of helicity projection operators to project out the desired left- or right-handed spinors; and second, by plugging explicit expressions for these spinors directly into our formula for the amplitude $\mathcal{M}$. Throughout this massless. (The calculation can be done for lower energy, but it is much more difficult and no more instructive.)†

Our starting point for both methods of calculating the polarized cross section is the amplitude

$$i\mathcal{M}\big(e^-(p)e^+(p') \to \mu^-(k)\mu^+(k')\big) = \frac{ie^2}{q^2}\Big(\bar{v}(p')\gamma^\mu u(p)\Big)\Big(\bar{u}(k)\gamma_\mu v(k')\Big). \quad (5.1)$$

We would like to use the spin sum identities to write the squared amplitude in terms of traces as before, even though we now want to consider only one set of polarizations at a time. To do this, we note that for massless fermions, the matrices

$$\frac{1+\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \frac{1-\gamma^5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (5.17)$$

are *projection operators* onto right- and left-handed spinors, respectively. Thus if in (5.1) we make the replacement

$$\bar{v}(p')\gamma^\mu u(p) \longrightarrow \bar{v}(p')\gamma^\mu\Big(\frac{1+\gamma^5}{2}\Big)u(p),$$

the amplitude for a right-handed electron is unchanged while that for a left-handed electron becomes zero. Note that since

$$\bar{v}(p')\gamma^\mu\Big(\frac{1+\gamma^5}{2}\Big)u(p) = v^\dagger(p')\Big(\frac{1+\gamma^5}{2}\Big)\gamma^0\gamma^\mu u(p), \quad (5.18)$$

this same replacement imposes the requirement that $v(p')$ also be a right-handed spinor. Recall from Section 3.5, however, that the right-handed spinor $v(p')$ corresponds to a *left*-handed positron. Thus we see that the annihilation amplitude vanishes when both the electron and the positron are right-handed. In general, the amplitude vanishes (in the massless limit) unless the electron and positron have opposite helicity, or equivalently, unless their spinors have the same helicity.

Having inserted this projection operator, we are now free to sum over the electron and positron spins in the squared amplitude; of the four terms in the sum, only one (the one we want) is nonzero. The electron half of $|\mathcal{M}|^2$, for a right-handed electron and a left-handed positron, is then

$$\begin{aligned}\sum_{\text{spins}}\Big|\bar{v}(p')\gamma^\mu\Big(\frac{1+\gamma^5}{2}\Big)u(p)\Big|^2 &= \sum_{\text{spins}}\bar{v}(p')\gamma^\mu\Big(\frac{1+\gamma^5}{2}\Big)u(p)\,\bar{u}(p)\gamma^\nu\Big(\frac{1+\gamma^5}{2}\Big)v(p') \\ &= \operatorname{tr}\Big[\not{p}'\gamma^\mu\Big(\frac{1+\gamma^5}{2}\Big)\not{p}\gamma^\nu\Big(\frac{1+\gamma^5}{2}\Big)\Big] \\ &= \operatorname{tr}\Big[\not{p}'\gamma^\mu\not{p}\gamma^\nu\Big(\frac{1+\gamma^5}{2}\Big)\Big]\end{aligned}$$

$$= 2\big(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}p\cdot p' - i\epsilon^{\alpha\mu\beta\nu}p'_\alpha p_\beta\big). \quad (5.19)$$

The indices in this expression are to be dotted into those of the muon half of the squared amplitude. For a right-handed μ^- and a left-handed μ^+, an identical calculation yields

$$\sum_{\text{spins}}\Big|\bar{u}(k)\gamma_\mu\Big(\frac{1+\gamma^5}{2}\Big)v(k')\Big|^2 = 2\big(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}k\cdot k' - i\epsilon_{\rho\mu\sigma\nu}k^\rho k'^\sigma\big). \quad (5.20)$$

Dotting (5.19) into (5.20), we find that the squared matrix element for $e^-_R e^+_L \to \mu^-_R\mu^+_L$ in the center-of-mass frame is

$$\begin{aligned}|\mathcal{M}|^2 &= \frac{4e^4}{q^4}\Big[2(p\cdot k)(p'\cdot k') + 2(p\cdot k')(p'\cdot k) - \epsilon^{\alpha\mu\beta\nu}\epsilon_{\rho\mu\sigma\nu}p'_\alpha p_\beta k^\rho k'^\sigma\Big] \\ &= \frac{8e^4}{q^4}\Big[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) - (p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k)\Big] \\ &= \frac{16e^4}{q^4}(p\cdot k')(p'\cdot k) \\ &= e^4(1+\cos\theta)^2. \end{aligned} \quad (5.21)$$

Plugging this result into (4.85) gives the differential cross section,

$$\frac{d\sigma}{d\Omega}(e^-_R e^+_L \to \mu^-_R\mu^+_L) = \frac{\alpha^2}{4E^2_{\text{cm}}}(1+\cos\theta)^2. \quad (5.22)$$

There is no need to repeat the entire calculation to obtain the other three nonvanishing helicity amplitudes. For example, the squared amplitude for $e^-_R e^+_L \to \mu^-_L\mu^+_R$ is identical to (5.20) but with γ^5 replaced by $-\gamma^5$ on the left-hand side, and thus $\epsilon_{\rho\mu\sigma\nu}$ replaced by $-\epsilon_{\rho\mu\sigma\nu}$ on the right-hand side. Propagating this sign though (5.21), we easily see that

$$\frac{d\sigma}{d\Omega}(e^-_R e^+_L \to \mu^-_L\mu^+_R) = \frac{\alpha^2}{4E^2_{\text{cm}}}(1-\cos\theta)^2. \quad (5.23)$$

Similarly,

$$\begin{aligned}\frac{d\sigma}{d\Omega}(e^-_L e^+_R \to \mu^-_R\mu^+_L) &= \frac{\alpha^2}{4E^2_{\text{cm}}}(1-\cos\theta)^2; \\ \frac{d\sigma}{d\Omega}(e^-_L e^+_R \to \mu^-_L\mu^+_R) &= \frac{\alpha^2}{4E^2_{\text{cm}}}(1+\cos\theta)^2.\end{aligned} \quad (5.24)$$

(These two results actually follow from the previous two by parity invariance.) The other twelve helicity cross sections (for instance, $e^-_L e^+_R \to \mu^-_L\mu^+_L$) are zero, as we saw from Eq. (5.18). Adding up all sixteen contributions, and dividing by 4 to average over the electron and positron spins, we recover the unpolarized

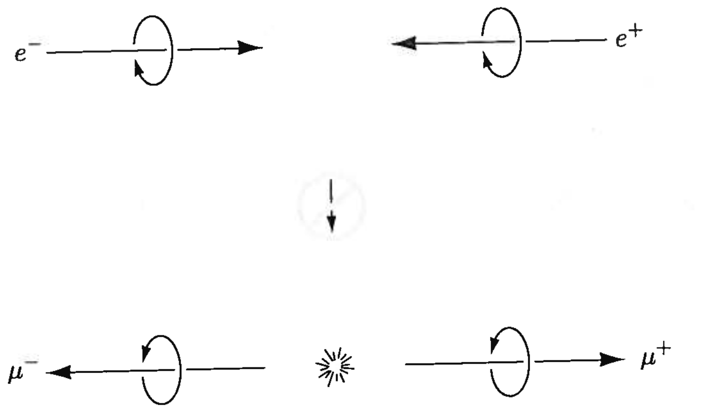


Figure 5.4. Conservation of angular momentum requires that if the z-component of angular momentum is measured, it must have the same value as initially.

Note that the cross section (5.22) for $e^-_R e^+_L \to \mu^-_R \mu^+_L$ vanishes at $\theta = 180°$. This is just what we would expect, since for $\theta = 180°$, the total angular momentum of the final state is opposite to that of the initial state (see Figure 5.4).

This completes our first calculation of the polarized $e^+e^- \to \mu^+\mu^-$ cross sections. We will now redo the calculation in a manner that is more straightforward, more enlightening, and no more difficult. We will calculate the amplitude $\mathcal{M}$ (rather than the squared amplitude) directly, using explicit values for the spinors and γ matrices. This method does have its drawbacks: It forces us to specialize to a particular frame of reference much sooner, so manifest Lorentz invariance is lost. More pragmatically, it is very cumbersome except in the nonrelativistic and ultra-relativistic limits.

Consider again the amplitude

$$\mathcal{M} = \frac{e^2}{q^2}\Big(\bar{v}(p')\gamma^\mu u(p)\Big)\Big(\bar{u}(k)\gamma_\mu v(k')\Big). \tag{5.25}$$

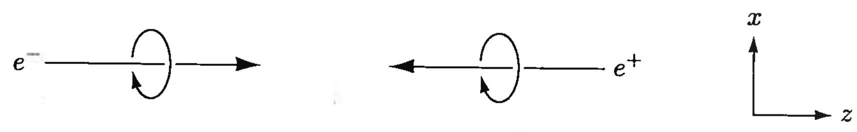
In the high-energy limit, our general expressions for Dirac spinors become

$$\begin{aligned} u(p) &= \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix} \underset{E\to\infty}{\longrightarrow} \sqrt{2E}\begin{pmatrix} \frac{1}{2}(1-\hat{p}\cdot\boldsymbol{\sigma})\xi \\ \frac{1}{2}(1+\hat{p}\cdot\boldsymbol{\sigma})\xi \end{pmatrix}; \\ v(p) &= \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ -\sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix} \underset{E\to\infty}{\longrightarrow} \sqrt{2E}\begin{pmatrix} \frac{1}{2}(1-\hat{p}\cdot\boldsymbol{\sigma})\xi \\ -\frac{1}{2}(1+\hat{p}\cdot\boldsymbol{\sigma})\xi \end{pmatrix}. \end{aligned} \tag{5.26}$$

A right-handed spinor satisfies $(\hat{p}\cdot\boldsymbol{\sigma})\xi = +\xi$, while a left-handed spinor has $(\hat{p}\cdot\boldsymbol{\sigma})\xi = -\xi$. (Remember once again that for antiparticles, the handedness of the spinor is the opposite of the handedness of the particle.) We must evaluate expressions of the form $\bar{v}\gamma^\mu u$, so we need

Thus we see explicitly that the amplitude is zero when one of the spinors is left-handed and the other is right-handed. In the language of Chapter 1, the Clebsch-Gordan coefficients that couple the vector photon to the product of such spinors are zero; those coefficients are just the off-block-diagonal elements of the matrix $\gamma^0\gamma^\mu$ (in the chiral representation).

Let us choose p and p' to be in the $\pm z$-directions, and first consider the case where the electron is right-handed and the positron is left-handed:



Thus for the electron we have $\xi = \binom{1}{0}$, corresponding to spin up in the z-direction, while for the positron we have $\xi = \binom{0}{1}$, also corresponding to (physical) spin up in the z-directon. Both particles have $(\hat{p}\cdot\boldsymbol{\sigma})\xi = +\xi$, so the spinors are

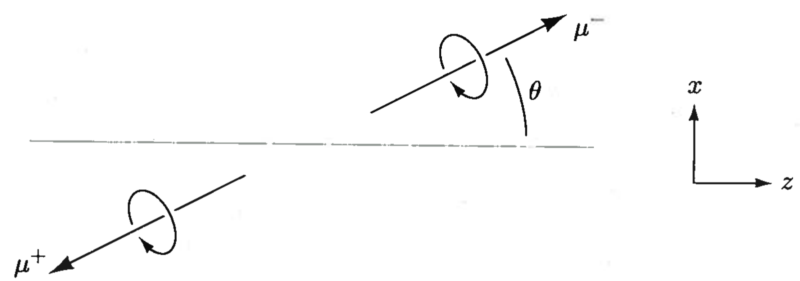
$$u(p) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}; \qquad v(p') = \sqrt{2E}\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}. \tag{5.28}$$

The electron half of the matrix element is therefore

$$\bar{v}(p')\gamma^\mu u(p) = 2E\,(0,\ -1)\sigma^\mu\begin{pmatrix}1\\0\end{pmatrix} = -2E\,(0,1,i,0). \tag{5.29}$$

We can interpret this expression by saying that the virtual photon has circular polarization in the $+z$-direction; its polarization vector is $\boldsymbol{\epsilon}_+ = (1/\sqrt{2})(\hat{x}+i\hat{y})$.

Next we must calculate the muon half of the matrix element. Let the μ^- be emitted at an angle θ to the z-axis, and consider first the case where it is right-handed (and the μ^+ is therefore left-handed):



To calculate $\bar{u}(k)\gamma^\mu v(k')$ we could go back to expressions (5.26), but then it would be necessary to find the correct spinors ξ corresponding to polarization along the muon momentum. It is much easier to use a trick: Since any expres-

(5.29). Rotating that vector by an angle θ in the xz-plane, we find

$$\begin{aligned}\bar{u}(k)\gamma^\mu v(k') &= \left[\bar{v}(k')\gamma^\mu u(k)\right]^* \\ &= \left[-2E\,(0,\cos\theta, i, -\sin\theta)\right]^* \\ &= -2E\,(0,\cos\theta,-i,-\sin\theta).\end{aligned} \tag{5.30}$$

This vector can also be interpreted as the polarization of the virtual photon; when it has a nonzero overlap with (5.29), we get a nonzero amplitude. Plugging (5.29) and (5.30) into (5.25), we see that the amplitude is

$$\mathcal{M}(e^-_R e^+_L \to \mu^-_R \mu^+_L) = \frac{e^2}{q^2}(2E)^2(-\cos\theta - 1) = -e^2(1+\cos\theta), \tag{5.31}$$

in agreement with (1.6), and also with (5.21). The differential cross section for this set of helicities can now be obtained in the same way as above, yielding (5.22).

We can calculate the other three nonvanishing helicity amplitudes in an analogous manner. For a left-handed electron and a right-handed positron, we easily find

$$\bar{v}(p')\gamma^\mu u(p) = -2E\,(0,1,-i,0) \equiv -2E\cdot\sqrt{2}\,\epsilon^\mu_-.$$

Perform a rotation to get the vector corresponding to a left-handed μ^- and a right-handed μ^+:

$$\bar{u}(k)\gamma^\mu v(k') = -2E\,(0,\cos\theta, i, -\sin\theta).$$

Putting the pieces together in various ways yields the remaining amplitudes,

$$\begin{aligned}\mathcal{M}(e^-_L e^+_R \to \mu^-_L \mu^+_R) &= -e^2(1+\cos\theta); \\ \mathcal{M}(e^-_R e^+_L \to \mu^-_L \mu^+_R) = \mathcal{M}(e^-_L e^+_R \to \mu^-_R \mu^+_L) &= -e^2(1-\cos\theta).\end{aligned} \tag{5.32}$$

5.3 $e^+e^- \to \mu^+\mu^-$: Nonrelativistic Limit

Now let us go to the other end of the energy spectrum, and discuss the reaction $e^+e^- \to \mu^+\mu^-$ in the extreme nonrelativistic limit. When E is barely larger than m_μ, our previous result (5.12) for the unpolarized differential cross section becomes

$$\frac{d\sigma}{d\Omega} \underset{|\mathbf{k}|\to 0}{\longrightarrow} \frac{\alpha^2}{2E^2_{\text{cm}}}\sqrt{1-\frac{m^2_\mu}{E^2}} = \frac{\alpha^2}{2E^2_{\text{cm}}}\frac{|\mathbf{k}|}{E}. \tag{5.33}$$

We can recover this result, and also learn something about the spin dependence of the reaction, by evaluating the amplitude with explicit spinors. Once again we begin with the matrix element

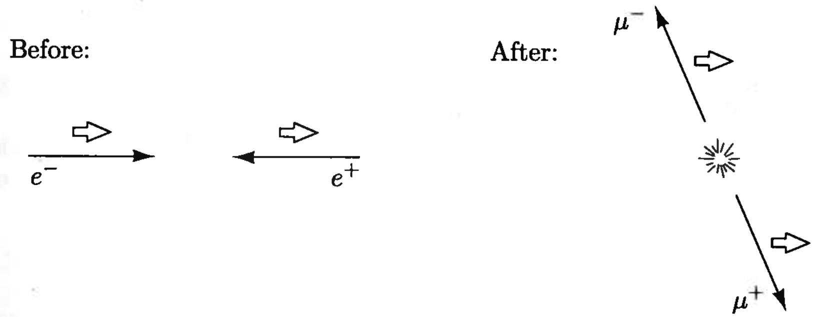


Figure 5.5. In the nonrelativistic limit the total spin of the system is conserved, and thus the muons are produced with both spins up along the z-axis.

The electron and positron are still very relativistic, so this expression will be simplest if we choose them to have definite helicity. Let the electron be right-handed, moving in the $+z$-direction, and the positron be left-handed, moving in the $-z$-direction. Then from Eq. (5.29) we have

$$\bar{v}(p')\gamma^\mu u(p) = -2E\,(0,1,i,0). \tag{5.34}$$

In the other half of the matrix element we should use the nonrelativistic expressions

$$u(k) = \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}, \qquad v(k') = \sqrt{m}\begin{pmatrix}\xi'\\ -\xi'\end{pmatrix}. \tag{5.35}$$

Keep in mind, in the discussion of this section, that the spinor ξ' gives the flipped spin of the antiparticle. Leaving the muon spinors ξ and ξ' undetermined for now, we can easily compute

$$\begin{aligned}\bar{u}(k)\gamma^\mu v(k') &= m(\xi^\dagger,\xi^\dagger)\begin{pmatrix}\bar\sigma^\mu & 0\\ 0 & \sigma^\mu\end{pmatrix}\begin{pmatrix}\xi'\\ -\xi'\end{pmatrix} \\ &= \begin{cases}0 & \text{for } \mu = 0,\\ -2m\xi^\dagger\sigma^i\xi' & \text{for } \mu = i.\end{cases}\end{aligned} \tag{5.36}$$

To evaluate $\mathcal{M}$, we simply dot (5.34) into (5.36) and multiply by $e^2/q^2 = e^2/4m^2$. The result is

$$\mathcal{M}(e^-_R e^+_L \to \mu^+\mu^-) = -2e^2\xi^\dagger\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\xi'. \tag{5.37}$$

Since there is no angular dependence in this expression, the muons are equally likely to come out in any direction. More precisely, they are emitted in an s-wave; their orbital angular momentum is zero. Angular momentum conservation therefore requires that the total spin of the final state equal 1, and indeed the matrix product gives zero unless both the muon and the antimuon

To find the total rate for this process, we sum over muon spins to obtain $\mathcal{M}^2 = 4e^4$, which yields the cross section

$$\frac{d\sigma}{d\Omega}(e^-_R e^+_L \to \mu^+\mu^-) = \frac{\alpha^2}{E^2_{\rm cm}}\frac{|\mathbf{k}|}{E}. \tag{5.38}$$

The same expression holds for a left-handed electron and a right-handed positron. Thus the spin-averaged cross section is just $2 \cdot (1/4)$ times this expression, in agreement with (5.33).

Bound States

Until now we have considered the initial and final states of scattering processes to be states of isolated single particles. Very close to threshold, however, the Coulomb attraction of the muons should become an important effect. Just below threshold, we can still form $\mu^+\mu^-$ pairs in electromagnetic bound states.

The treatment of bound states in quantum field theory is a rich and complex subject, but one that lies mainly beyond the scope of this book.‡ Fortunately, many of the familiar bound systems in Nature can be treated (at least to a good first approximation) as nonrelativistic systems, in which the internal motions are slow. The process of creating the constituent particles out of the vacuum is still a relativistic effect, requiring quantum field theory for its proper description. In this section we will develop a formalism for computing the amplitudes for creation and annihilation of two-particle, nonrelativistic bound states. We begin with a computation of the cross section for producing a $\mu^+\mu^-$ bound state in e^+e^- annihilation.

Consider first the case where the spins of the electron and positron both point up along the z-axis. From the preceding discussion we know that the resulting muons both have spin up, so the only type of bound state we can produce will have total spin 1, also pointing up. The amplitude for producing free muons in this configuration is

$$\mathcal{M}(\uparrow\uparrow \to \mathbf{k}_1\uparrow, \mathbf{k}_2\uparrow) = -2e^2, \tag{5.39}$$

independent of the momenta (which we now call $\mathbf{k}_1$ and $\mathbf{k}_2$) of the muons.

Next we need to know how to write a bound state in terms of free-particle states. For a general two-body system with equal constituent masses, the center-of-mass and relative coordinates are

$$\mathbf{R} = \tfrac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2), \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2. \tag{5.40}$$

These have conjugate momenta

$$\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2, \qquad \mathbf{k} = \tfrac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2). \tag{5.41}$$

The total momentum $\mathbf{K}$ is zero in the center-of-mass frame. If we know the force between the particles (for $\mu^+\mu^-$, it is just the Coulomb force), we can solve the nonrelativistic Schrödinger equation to find the Schrödinger wavefunction, $\psi(\mathbf{r})$. The bound state is just a linear superposition of free states of definite $\mathbf{r}$ or $\mathbf{k}$, weighted by this wavefunction. For our purposes it is more convenient to build this superposition in momentum space, using the Fourier transform of $\psi(\mathbf{r})$:

$$\widetilde{\psi}(\mathbf{k}) = \int d^3x\, e^{i\mathbf{k}\cdot\mathbf{r}}\psi(\mathbf{r}); \qquad \int \frac{d^3k}{(2\pi)^3}\left|\widetilde{\psi}(\mathbf{k})\right|^2 = 1. \tag{5.42}$$

If $\psi(\mathbf{r})$ is normalized conventionally, $\widetilde{\psi}(\mathbf{k})$ gives the amplitude for finding a particular value of $\mathbf{k}$. An explicit expression for a bound state with mass $M \approx 2m$, momentum $\mathbf{K} = 0$, and spin 1 oriented up is then

$$|B\rangle = \sqrt{2M}\int \frac{d^3k}{(2\pi)^3}\,\widetilde{\psi}(\mathbf{k})\,\frac{1}{\sqrt{2m}}\frac{1}{\sqrt{2m}}\,|\mathbf{k}\uparrow, -\mathbf{k}\uparrow\rangle. \tag{5.43}$$

The factors of $(1/\sqrt{2m})$ convert our relativistically normalized free-particle states so that their integral with $\widetilde{\psi}(\mathbf{k})$ is a state of norm 1. (The factors should involve $\sqrt{2E_{\pm\mathbf{k}}}$, but for a nonrelativistic bound state, $|\mathbf{k}| \ll m$.) The outside factor of $\sqrt{2M}$ converts back to the relativistic normalization assumed by our formula for cross sections. These normalization factors could easily be modified to describe a bound state with nonzero total momentum $\mathbf{K}$.

Given this expression for the bound state, we can immediately write down the amplitude for its production:

$$\mathcal{M}(\uparrow\uparrow \to B) = \sqrt{2M}\int \frac{d^3k}{(2\pi)^3}\,\widetilde{\psi}^*(\mathbf{k})\,\frac{1}{\sqrt{2m}}\frac{1}{\sqrt{2m}}\,\mathcal{M}(\uparrow\uparrow \to \mathbf{k}\uparrow, -\mathbf{k}\uparrow). \tag{5.44}$$

Since the free-state amplitude from (5.39) is independent of the momenta of the muons, the integral over $\mathbf{k}$ gives $\psi^*(0)$, the position-space wavefunction evaluated at the origin. It is quite natural that the amplitude for creation of a two-particle state from a pointlike virtual photon should be proportional to the value of the wavefunction at zero separation. Assembling the pieces, we find that the amplitude is simply

$$\mathcal{M}(\uparrow\uparrow \to B) = \sqrt{\frac{2}{M}}(-2e^2)\psi^*(0). \tag{5.45}$$

In a moment we will compute the cross section from this amplitude. First, however, let us generalize this discussion to treat bound states with more general spin configurations. The analysis leading up to (5.37) will cast any S-matrix element for the production of nonrelativistic fermions with momenta $\mathbf{k}$ and $-\mathbf{k}$ into the form of a spin matrix element

$$i\mathcal{M}(\text{something} \to \mathbf{k}, \mathbf{k}') = \xi^\dagger[\Gamma(\mathbf{k})]\xi', \tag{5.46}$$

where $\Gamma(\mathbf{k})$ is some 2×2 matrix. We now must replace the spinors with a nor-

Exercise 3

The substitution into the kinetic term of the scalar field yields

$$
\begin{aligned}
(D_\mu H)^\dagger D^\mu H &= \left(\begin{pmatrix}0 & \partial_\mu h\end{pmatrix} - \begin{pmatrix}0 & v+h\end{pmatrix}\left[\frac{i}{2}\sigma^I W^I_\mu + \frac{i}{2}B_\mu\right]\right)\left(\begin{pmatrix}0\\ \partial^\mu h\end{pmatrix} + \left[\frac{i}{2}\sigma^I W^I_\mu + \frac{i}{2}B_\mu\right]\begin{pmatrix}0\\ v+h\end{pmatrix}\right)\\
&= (\partial_\mu h)^2 + \frac{1}{4}\begin{pmatrix}0 & v+h\end{pmatrix}\left(W^I_\mu\sigma^I + B_\mu\right)\left(W^{J\,\mu}\sigma^J + B^\mu\right)\begin{pmatrix}0\\ v+h\end{pmatrix}\\
&= (\partial_\mu h)^2 + \frac{(v+h)^2}{4}\left[(W^1_\mu)^2 + (W^2_\mu)^2 + (-W^3_\mu + B_\mu)^2\right] .
\end{aligned}
$$

One quick way to show the last equality without computing every Pauli matrix product is to symmetrize the product as follows

$$
W^I_\mu W^{\mu\,J}\sigma^I\sigma^J = \frac{1}{2}W^I_\mu W^{\mu\,J}\{\sigma^I,\sigma^J\} = W^I_\mu W^{\mu\,J}\delta^{IJ} .
$$

Then, we want to rescale the gauge fields to have canonically normalized kinetic terms. This means doing

$$
W^I_\mu \to g W^I_\mu , \qquad B_\mu \to g_Y B_\mu .
$$

At this point the lagrangian has become

$$
\mathcal{L} = -\frac{1}{4}W^I_{\mu\nu}W^{I\,\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + (\partial_\mu h)^2 + \frac{(v+h)^2}{4}\left[g^2(W^1_\mu)^2 + g^2(W^2_\mu)^2 + (-gW^3_\mu + g_Y B_\mu)^2\right] - m^2(v+h)^2 + \frac{\lambda}{2}(v+h)^4 .
$$

Notice that it would be natural to also rescale the neutral scalar h by $h \to h/\sqrt{2}$ to make the third term canonically normalized, but we will not do it here since it does not affect the vector mass terms.

The fourth term describes some interactions between the h and vector fields. Ignoring these, we focus on the vector mass terms

$$
\frac{v^2}{4}\left[g^2(W^1_\mu)^2 + g^2(W^2_\mu)^2 + (-gW^3_\mu + g_Y B_\mu)^2\right] = \frac{v^2}{4}\begin{pmatrix}W^1_\mu & W^2_\mu & W^3_\mu & B_\mu\end{pmatrix}\begin{pmatrix} g^2 & & & \\ & g^2 & & \\ & & g^2 & -gg_Y \\ & & -gg_Y & g_Y^2\end{pmatrix}\begin{pmatrix}W^{1\mu}\\ W^{2\mu}\\ W^{3\mu}\\ B^\mu\end{pmatrix} .
$$

We have found the mass matrix of vector fields

$$
M_v = \frac{v^2}{2}\begin{pmatrix} g^2 & & & \\ & g^2 & & \\ & & g^2 & -gg_Y \\ & & -gg_Y & g_Y^2\end{pmatrix}
$$

which we now want to diagonalize through a unitary transformation to preserve normalization of the fields. It is already diagonal in the first two fields, but it is customary to define

$$
W^\pm_\mu = \frac{W^1_\mu \mp i W^2_\mu}{\sqrt{2}}
$$

because these will have a definite electric charge. The second block can be diagonalized by a rotation of the last two fields

$$
\begin{cases} Z_\mu = \cos\theta_W W^3_\mu - \sin\theta_W B_\mu \\ A_\mu = \sin\theta_W W^3_\mu + \cos\theta_W B_\mu \end{cases} , \qquad \sin\theta_W = \frac{g_Y}{\sqrt{g^2 + {g_Y}^2}}
$$

The vector kinetic and mass terms of the lagrangian can now be rewritten as

$$
-\frac{1}{2}W^+_{\mu\nu}W^{-\,\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} + m_W^2 W^+_\mu W^{-\,\mu} + \frac{m_Z^2}{2}Z_\mu Z^\mu
$$

where we defined the masses $m_W^2 = \frac{g^2v^2}{2}$, $m_Z^2 = \frac{(g^2+{g_Y}^2)v^2}{2}$. The field A_μ is massless and is interpeted as the photon field.