Thus the Mandelstam variables are

$$s = (p+p')^2 = (2E)^2 = E_{\rm cm}^2;$$
$$t = (k-p)^2 = -p^2\sin^2\theta - p^2(\cos\theta - 1)^2 = -2p^2(1-\cos\theta); \qquad (5.72)$$
$$u = (k'-p)^2 = -p^2\sin^2\theta - p^2(\cos\theta+1)^2 = -2p^2(1+\cos\theta).$$

Thus we see that $t \to 0$ as $\theta \to 0$, while $u \to 0$ as $\theta \to \pi$. (When the masses are not all equal, the limiting values of t and u will shift slightly.)

Note from (5.72) that when all four particles have mass m, the sum of the Mandelstam variables is $s+t+u = 4E^2 - 4p^2 = 4m^2$. This is a special case of a more general relation, which is often quite useful:

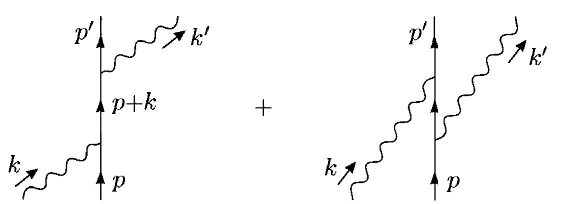
$$s + t + u = \sum_{i=1}^{4} m_i^2, \qquad (5.73)$$

where the sum runs over the four external particles. This identity is easy to prove by adding up the terms on the right-hand side of Eqs. (5.69), and applying momentum conservation in the form $(p+p'-k-k')^2 = 0$.

5.5 Compton Scattering

We now move on to consider a somewhat different QED process: *Compton scattering*, or $e^-\gamma \to e^-\gamma$. We will calculate the unpolarized cross section for this reaction, to lowest order in α. The calculation will employ all the machinery we have developed so far, including the Mandelstam variables of the previous section. We will also develop some new technology for dealing with external photons.

This is our first example of a calculation involving two diagrams:



As usual, the Feynman rules tell us exactly how to write down an expression for $\mathcal{M}$. Note that since the fermion portions of the two diagrams are identical, there is no relative minus sign between the two terms. Using $\epsilon_\nu(k)$ and $\epsilon^*_\mu(k')$ to denote the polarization vectors of the initial and final photons, we have

$$i\mathcal{M} = \bar{u}(p')(-ie\gamma^\mu)\epsilon^*_\mu(k')\frac{i(\not p + \not k + m)}{(p+k)^2 - m^2}(-ie\gamma^\nu)\epsilon_\nu(k)u(p)$$
$$+\, \bar{u}(p')(-ie\gamma^\nu)\epsilon_\nu(k)\frac{i(\not p - \not k' + m)}{(p-k')^2 - m^2}(-ie\gamma^\mu)\epsilon^*_\mu(k')u(p)$$

$$= -ie^2\epsilon^*_\mu(k')\epsilon_\nu(k)\,\bar{u}(p')\left[\frac{\gamma^\mu(\not p+\not k+m)\gamma^\nu}{(p+k)^2-m^2} + \frac{\gamma^\nu(\not p-\not k'+m)\gamma^\mu}{(p-k')^2-m^2}\right]u(p).$$

We can make a few simplifications before squaring this expression. Since $p^2 = m^2$ and $k^2 = 0$, the denominators of the propagators are

$$(p+k)^2 - m^2 = 2p\cdot k \qquad \text{and} \qquad (p-k')^2 - m^2 = -2p\cdot k'.$$

To simplify the numerators, we use a bit of Dirac algebra:

$$(\not p + m)\gamma^\nu u(p) = (2p^\nu - \gamma^\nu \not p + \gamma^\nu m)u(p)$$
$$= 2p^\nu u(p) - \gamma^\nu(\not p - m)u(p)$$
$$= 2p^\nu u(p).$$

Using this trick on the numerator of each propagator, we obtain

$$i\mathcal{M} = -ie^2\epsilon^*_\mu(k')\epsilon_\nu(k)\,\bar{u}(p')\left[\frac{\gamma^\mu \not k\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{-\gamma^\nu \not k'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\right]u(p). \qquad (5.74)$$

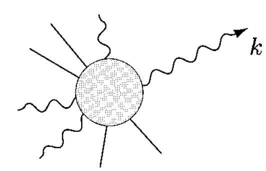
Photon Polarization Sums

The next step in the calculation will be to square this expression for $\mathcal{M}$ and sum (or average) over electron and photon polarization states. The sum over electron polarizations can be performed as before, using the identity $\Sigma u(p)\bar{u}(p) = \not p + m$. Fortunately, there is a similar trick for summing over photon polarization vectors. The correct prescription is to make the replacement

$$\sum_{\text{polarizations}} \epsilon^*_\mu \epsilon_\nu \longrightarrow -g_{\mu\nu}. \qquad (5.75)$$

The arrow indicates that this is not an actual equality. Nevertheless, the replacement is valid as long as both sides are dotted into the rest of the expression for a QED amplitude $\mathcal{M}$.

To derive this formula, let us consider an arbitrary QED process involving an external photon with momentum k:



$$= i\mathcal{M}(k) \equiv i\mathcal{M}^\mu(k)\epsilon^*_\mu(k). \qquad (5.76)$$

Since the amplitude always contains $\epsilon^*_\mu(k)$, we have extracted this factor and defined $\mathcal{M}^\mu(k)$ to be the rest of the amplitude $\mathcal{M}$. The cross section will be proportional to

$$\sum_\epsilon \left|\epsilon^*_\mu(k)\mathcal{M}^\mu(k)\right|^2 = \sum_\epsilon \epsilon^*_\mu \epsilon_\nu\, \mathcal{M}^\mu(k)\mathcal{M}^{\nu *}(k).$$

For simplicity, we orient k in the 3-direction: $k^\mu = (k, 0, 0, k)$. Then the two transverse polarization vectors, over which we are summing, can be chosen to be

$$\epsilon_1^\mu = (0,1,0,0); \qquad \epsilon_2^\mu = (0,0,1,0).$$

With these conventions, we have

$$\sum_\epsilon \left|\epsilon_\mu^*(k)\mathcal{M}^\mu(k)\right|^2 = \left|\mathcal{M}^1(k)\right|^2 + \left|\mathcal{M}^2(k)\right|^2. \tag{5.77}$$

Now recall from Chapter 4 that external photons are created by the interaction term $\int d^4x\, e j^\mu A_\mu$, where $j^\mu = \bar\psi\gamma^\mu\psi$ is the Dirac vector current. Therefore we expect $\mathcal{M}^\mu(k)$ to be given by a matrix element of the Heisenberg field j^μ:

$$\mathcal{M}^\mu(k) = \int d^4x\, e^{ik\cdot x} \langle f|\, j^\mu(x)\,|i\rangle\,, \tag{5.78}$$

where the initial and final states include all particles except the photon in question.

From the classical equations of motion, we know that the current j^μ is conserved: $\partial_\mu j^\mu(x) = 0$. Provided that this property still holds in the quantum theory, we can dot k_μ into (5.78) to obtain

$$k_\mu \mathcal{M}^\mu(k) = 0. \tag{5.79}$$

The amplitude $\mathcal{M}$ vanishes when the polarization vector $\epsilon_\mu(k)$ is replaced by k_μ. This famous relation is known as the *Ward identity*. It is essentially a statement of current conservation, which is a consequence of the gauge symmetry (4.6) of QED. We will give a formal proof of the Ward identity in Section 7.4, and discuss a number of subtle points skimmed over in this quick "derivation".

It is useful to check explicitly that the Compton amplitude given in (5.74) obeys the Ward identity. To do this, replace $\epsilon_\nu(k)$ by k_ν or $\epsilon_\mu^*(k')$ by k'_μ, and manipulate the Dirac matrix products. In either case (after a bit of algebra) the terms from the two diagrams cancel each other to give zero.

Returning to our derivation of the polarization sum formula (5.75), we note that for $k^\mu = (k,0,0,k)$, the Ward identity takes the form

$$k\mathcal{M}^0(k) - k\mathcal{M}^3(k) = 0. \tag{5.80}$$

Thus $\mathcal{M}^0 = \mathcal{M}^3$, and we have

$$\begin{aligned}\sum_\epsilon \epsilon_\mu^*\epsilon_\nu\, \mathcal{M}^\mu(k)\mathcal{M}^{\nu*}(k) &= |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 \\ &= |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 + |\mathcal{M}^3|^2 - |\mathcal{M}^0|^2 \\ &= -g_{\mu\nu}\,\mathcal{M}^\mu(k)\mathcal{M}^{\nu*}(k).\end{aligned}$$

That is, we may sum over external photon polarizations by replacing $\sum \epsilon_\mu^*\epsilon_\nu$ with $-g_{\mu\nu}$.

Note that this proves (pending our general proof of the Ward identity) that the unphysical timelike and longitudinal photons can be consistently omitted from QED calculations, since in any event the squared amplitudes for producing these states cancel to give zero total probability. The negative norm of the timelike photon state, a property that troubled us in the discussion after Eq. (4.132), plays an essential role in this cancellation.

The Klein-Nishina Formula

The rest of the computation of the Compton scattering cross section is straightforward, although it helps to be somewhat organized. We want to average the squared amplitude over the initial electron and photon polarizations, and sum over the final electron and photon polarizations. Starting with expression (5.74) for $\mathcal{M}$, we find

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 &= \frac{e^4}{4} g_{\mu\rho}\, g_{\nu\sigma}\cdot \mathrm{tr}\Big\{(\not p'+m)\Big[\frac{\gamma^\mu \not k\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{\gamma^\nu \not k'\gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'}\Big] \\ &\qquad\cdot(\not p+m)\Big[\frac{\gamma^\sigma \not k\gamma^\rho + 2\gamma^\rho p^\sigma}{2p\cdot k} + \frac{\gamma^\rho \not k'\gamma^\sigma - 2\gamma^\sigma p^\rho}{2p\cdot k'}\Big]\Big\} \\ &\equiv \frac{e^4}{4}\Big[\frac{\mathbf{I}}{(2p\cdot k)^2} + \frac{\mathbf{II}}{(2p\cdot k)(2p\cdot k')} + \frac{\mathbf{III}}{(2p\cdot k')(2p\cdot k)} + \frac{\mathbf{IV}}{(2p\cdot k')^2}\Big],\end{aligned} \tag{5.81}$$

where **I**, **II**, **III**, and **IV** are complicated traces. Note that **IV** is the same as **I** if we replace k with $-k'$. Also, since we can reverse the order of the γ matrices inside a trace (Eq. (5.7)), we see that **II** = **III**. Thus we must work only to compute **I** and **II**.

The first of the traces is

$$\mathbf{I} = \mathrm{tr}\big[(\not p' + m)(\gamma^\mu \not k\gamma^\nu + 2\gamma^\mu p^\nu)(\not p + m)(\gamma_\nu \not k\gamma_\mu + 2\gamma_\mu p_\nu)\big].$$

There are 16 terms inside the trace, but half contain an odd number of γ matrices and therefore vanish. We must now evaluate the other eight terms, one at a time. For example,

$$\begin{aligned}\mathrm{tr}\big[\not p'\gamma^\mu \not k\gamma^\nu \not p\gamma_\nu \not k\gamma_\mu\big] &= \mathrm{tr}\big[(-2\not p')\not k(-2\not p)\not k\big] \\ &= \mathrm{tr}\big[4\not p'\not k(2p\cdot k - \not k\not p)\big] \\ &= 8p\cdot k\ \mathrm{tr}\big[\not p'\not k\big] \\ &= 32(p\cdot k)(p'\cdot k).\end{aligned}$$

By similar use of the contraction identities (5.8) and (5.9), and other Dirac algebra such as $\not p\not p = p^2 = m^2$, each term in **I** can be reduced to a trace of no more than two γ matrices. When the smoke clears, we find

$$\mathbf{I} = 16\big(4m^4 - 2m^2 p\cdot p' + 4m^2 p\cdot k - 2m^2 p'\cdot k + 2(p\cdot k)(p'\cdot k)\big). \tag{5.82}$$

Although it is not obvious, this expression can be simplified further. To see how, introduce the Mandelstam variables:

$$\begin{aligned} s &= (p+k)^2 = 2p\cdot k + m^2 = 2p'\cdot k' + m^2; \\ t &= (p'-p)^2 = -2p\cdot p' + 2m^2 = -2k\cdot k'; \\ u &= (k'-p)^2 = -2k'\cdot p + m^2 = -2k\cdot p' + m^2. \end{aligned} \tag{5.83}$$

Recall from (5.73) that momentum conservation implies $s+t+u = 2m^2$. Writing everything in terms of s, t, and u, and using this identity, we eventually obtain

$$\mathbf{I} = 16\big(2m^4 + m^2(s-m^2) - \tfrac{1}{2}(s-m^2)(u-m^2)\big). \tag{5.84}$$

Sending $k \leftrightarrow -k'$, we can immediately write

$$\mathbf{IV} = 16\big(2m^4 + m^2(u-m^2) - \tfrac{1}{2}(s-m^2)(u-m^2)\big). \tag{5.85}$$

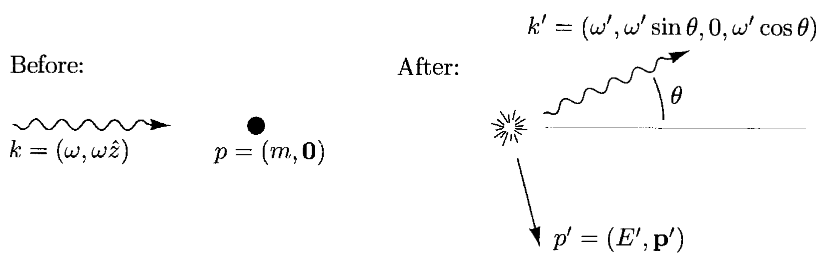
Evaluating the traces in the numerators **II** and **III** requires about the same amount of work as we have just done. The answer is

$$\mathbf{II} = \mathbf{III} = -8\big(4m^4 + m^2(s-m^2) + m^2(u-m^2)\big). \tag{5.86}$$

Putting together the pieces of the squared matrix element (5.81), and rewriting s and u in terms of $p\cdot k$ and $p\cdot k'$, we finally obtain

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4\left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right]. \tag{5.87}$$

To turn this expression into a cross section we must decide on a frame of reference and draw a picture of the kinematics. Compton scattering is most often analyzed in the "lab" frame, in which the electron is initially at rest:



We will express the cross section in terms of ω and θ. We can find ω', the energy of the final photon, using the following trick:

$$\begin{aligned} m^2 = (p')^2 &= (p+k-k')^2 = p^2 + 2p\cdot(k-k') - 2k\cdot k' \\ &= m^2 + 2m(\omega-\omega') - 2\omega\omega'(1-\cos\theta), \end{aligned}$$

hence,

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m}(1-\cos\theta). \tag{5.88}$$

The last line is Compton's formula for the shift in the photon wavelength. For our purposes, however, it is more useful to solve for ω':

$$\omega' = \frac{\omega}{1 + \dfrac{\omega}{m}(1-\cos\theta)}. \tag{5.89}$$

The phase space integral in this frame is

$$\begin{aligned} \int d\Pi_2 &= \int \frac{d^3k'}{(2\pi)^3}\frac{1}{2\omega'}\frac{d^3p'}{(2\pi)^3}\frac{1}{2E'}(2\pi)^4\delta^{(4)}(k'+p'-k-p) \\ &= \int \frac{(\omega')^2 d\omega'\, d\Omega}{(2\pi)^3}\frac{1}{4\omega' E'} \\ &\qquad \times 2\pi\,\delta\Big(\omega' + \sqrt{m^2+\omega^2+(\omega')^2-2\omega\omega'\cos\theta} - \omega - m\Big) \\ &= \int \frac{d\cos\theta}{2\pi}\frac{\omega'}{4E'}\frac{1}{\left|1 + \dfrac{\omega'-\omega\cos\theta}{E'}\right|} \\ &= \frac{1}{8\pi}\int d\cos\theta\,\frac{\omega'}{m+\omega(1-\cos\theta)} \\ &= \frac{1}{8\pi}\int d\cos\theta\,\frac{(\omega')^2}{\omega m}. \end{aligned} \tag{5.90}$$

Plugging everything into our general cross-section formula (4.79) and setting $|v_{\mathcal{A}} - v_{\mathcal{B}}| = 1$, we find

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2\omega}\frac{1}{2m}\cdot\frac{1}{8\pi}\frac{(\omega')^2}{\omega m}\cdot\left(\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\right).$$

To evaluate $|\mathcal{M}|^2$, we replace $p\cdot k = m\omega$ and $p\cdot k' = m\omega'$ in (5.87). The shortest way to write the final result is

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right], \tag{5.91}$$

where ω'/ω is given by (5.89). This is the (spin-averaged) *Klein-Nishina formula*, first derived in 1929.†

In the limit $\omega \to 0$ we see from (5.89) that $\omega'/\omega \to 1$, so the cross section becomes

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}(1+\cos^2\theta); \qquad \sigma_{\text{total}} = \frac{8\pi\alpha^2}{3m^2}. \tag{5.92}$$

This is the familiar Thomson cross section for scattering of classical electromagnetic radiation by a free electron.

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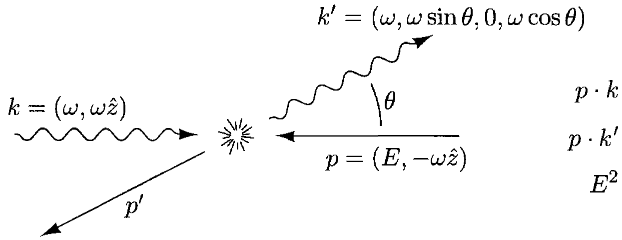
†O. Klein and Y. Nishina, *Z. Physik*, **52**, 853 (1929).

High-Energy Behavior

To analyze the high-energy behavior of the Compton scattering cross section, it is easiest to work in the center-of-mass frame. We can easily construct the differential cross section in this frame from the invariant expression (5.87). The kinematics of the reaction now looks like this:



$$p \cdot k = \omega(E + \omega)$$
$$p \cdot k' = \omega(E + \omega\cos\theta)$$
$$E^2 = \omega^2 + m^2$$

Plugging these values into (5.87), we see that for $\theta \approx \pi$, the term $p{\cdot}k/p{\cdot}k'$ becomes very large, while the other terms are all of $\mathcal{O}(1)$ or smaller. Thus for $E \gg m$ and $\theta \approx \pi$, we have

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 \approx 2e^4 \cdot \frac{p\cdot k}{p\cdot k'} = 2e^4 \cdot \frac{E+\omega}{E+\omega\cos\theta}. \tag{5.93}$$

The cross section in the CM frame is given by (4.84):

$$\begin{aligned}\frac{d\sigma}{d\cos\theta} &= \frac{1}{2}\cdot\frac{1}{2E}\cdot\frac{1}{2\omega}\cdot\frac{\omega}{(2\pi)4(E+\omega)}\cdot\frac{2e^4(E+\omega)}{E+\omega\cos\theta} \\ &\approx \frac{2\pi\alpha^2}{2m^2 + s(1+\cos\theta)}.\end{aligned} \tag{5.94}$$

Notice that, since $s \gg m^2$, the denominator of (5.94) almost vanishes when the photon is emitted in the backward direction ($\theta \approx \pi$). In fact, the electron mass m could be neglected completely in this formula if it were not necessary to cut off this singularity. To integrate over $\cos\theta$, we can drop the electron mass term if we supply an equivalent cutoff near $\theta = \pi$. In this way, we can approximate the total Compton scattering cross section by

$$\int_{-1}^{1} d(\cos\theta)\,\frac{d\sigma}{d\cos\theta} \approx \frac{2\pi\alpha^2}{s}\int_{-1+2m^2/s}^{1} d(\cos\theta)\,\frac{1}{(1+\cos\theta)}. \tag{5.95}$$

Thus, we find that the total cross section behaves at high energy as

$$\sigma_{\text{total}} = \frac{2\pi\alpha^2}{s}\log\Big(\frac{s}{m^2}\Big). \tag{5.96}$$

The main dependence α^2/s follows from dimensional analysis. But the singularity associated with backward scattering of photons leads to an enhancement by an extra logarithm of the energy.