Quantum Field Theory

Set 11: solutions

Exercise 1

Let us start from the matrix element squared

$$|\mathcal{M}|^2 = \lambda^4 \left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right]^2 .$$

Making use of the 2-body phase space in the case of final state particles with equal masses, we have

$$d\sigma = \frac{1}{4\sqrt{(p_a\cdot p_b)^2 - m^4}} |\mathcal{M}|^2 \beta \frac{d\cos\theta d\varphi}{64\pi^2},$$

where $\beta \equiv \sqrt{1-\frac{4m^2}{s}}$ is the velocity of the incoming particles in the center of mass frame. We have put 64 instead of 32 as a factor accounting for the identity of the two final state particles: thus we will integrate over all the phase space. Since we consider four identical masses we can further simplify the previous expression, because the flux factor is proportional to $1/\beta$:

$$d\sigma = |\mathcal{M}|^2 \frac{d\cos\theta d\varphi}{128\pi^2 s}.$$

Hence

$$\frac{d\sigma}{d\cos\theta} = \frac{\lambda^4}{64\pi s}\left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right]^2, \tag{1}$$

where we have integrated in $d\varphi$ because nothing depends on the azimuthal angle. Indeed one can show that, parametrizing momenta as

$$\begin{aligned} p_a &= \frac{\sqrt{s}}{2}(1,0,0,\beta), \\ p_b &= \frac{\sqrt{s}}{2}(1,0,0,-\beta), \\ p_c &= \frac{\sqrt{s}}{2}(1,0,\beta\sin\theta,\beta\cos\theta), \\ p_d &= \frac{\sqrt{s}}{2}(1,0,-\beta\sin\theta,-\beta\cos\theta), \end{aligned}$$

the independent Mandelstam invariants are simply

$$\begin{aligned} t \equiv (p_a-p_c)^2 = -\frac{s\beta^2}{2}(1-\cos\theta) = -2\left(\frac{s}{4}-m^2\right)(1-\cos\theta), \\ u \equiv (p_a-p_d)^2 = -\frac{s\beta^2}{2}(1+\cos\theta) = -2\left(\frac{s}{4}-m^2\right)(1+\cos\theta). \end{aligned}$$

Therefore the l.h.s of the differential cross section $\frac{d\sigma}{d\cos\theta}$ is only a function of θ.
Before performing the integration let us consider the high energy limit $s \gg m^2$. In this limit one would expect the total cross section not to depend on masses, and therefore, by dimensional analysis,

$$\sigma \simeq \frac{\lambda^4}{s^3}.$$

This behavior is wrong or, to say better, is not the dominant one in this limit. To see this, one can indeed expand the differential cross section in powers of m^2/s. Note that the mass acts as a regulator of the integral in $d\cos\theta$ (in the strict massless case there are non integrable singularities at $\cos\theta = \pm 1$ coming from the $t-$ and $u-$channels).

Therefore the expansion in powers of m^2/s at leading order takes place neglecting the mass for the $s-$channel (never singular) and in the *definitions* of t and u, but retaining it in the denominators of the $t-$ and $u-$channels. This yields

$$\sigma \simeq \frac{\lambda^4}{64\pi s}\int_{-1}^{1} d\cos\theta \left[\frac{1}{s} - \frac{2}{s(1-\cos\theta)+2m^2} - \frac{2}{s(1+\cos\theta)+2m^2}\right]^2 \tag{2}$$

To perform the above integration we can notice that in the massless limit the square of the first piece is finite, the interference terms of the $s-$channel with the others has a single pole, and the interference between the $t-$ and $u-$channels has two single poles: all these terms will diverge at most as $\log(m)$ in the small mass limit. Conversely, the squares of the $t-$ and $u-$channels have double poles, thus they will result in a m^{-2} dependence. By changing variable $y \to -y$ in one of the two contributions, it is immediate to notice that they are equal, so that the main contribution to the cross section will be

$$\sigma \simeq \frac{\lambda^4}{64\pi s} 2\int_{-1}^{1} d\cos\theta \left[\frac{2}{s(1+\cos\theta)+2m^2}\right]^2 = \frac{\lambda^4}{64\pi s}\left[\frac{4}{m^2 s}\right].$$

The leading behavior has a different power with respect to what we guessed by dimensional analysis.
One can understand this behavior in the following way: if we put the mass to zero the cross section diverges. This is because a massless particle can mediate long range (actually infinite range) interaction and therefore two particles interact even if they are far apart.

Now, we instead consider scattering at large angles $\theta \simeq \pi/2$. We are still in the relativistic limit, so we can still use the estimate (2), integrating on $\theta \in (\frac{\pi}{3}, \frac{2\pi}{3})$. These bounds are arbitrary, we just need to consider a range of angles close to the perpendicular. Thus we make the estimate $\cos\theta \simeq 0$, and we can neglect the mass in the denominators. We get

$$\sigma \simeq \frac{\lambda^4}{64\pi s}\int_{-1/2}^{1/2} d\cos\theta \left[\frac{3}{s}\right]^2 = \frac{9\lambda^4}{64\pi s^3}.$$

Here we have the result we guessed by numerical analysis. Indeed, this is the case of hard scattering, where the transverse momentum is large $p_\perp^2 \sim s \gg m^2$. This correspond to short distance interactions. In this regime, the mass can be totally neglected and plays no role as we have seen. This explains why this contribution matches the result of dimensional analysis.

In the opposite limit, the non-relativistic one, the differential cross section is perfectly finite over all the phase space, so it is possible to expand the integrand in Taylor series without any particular treatment (in the ultra relativistic limit we had to retain the mass in the propagators). At zeroth order in $s-4m^2$ one has $t=u=0$, thus

$$\frac{d\sigma}{d\cos\theta} \simeq \frac{\lambda^4}{64\pi\, 4m^2}\left[\frac{1}{4m^2-m^2} + \frac{1}{-m^2} + \frac{1}{-m^2}\right]^2 = \frac{25}{9}\times\frac{\lambda^4}{256\pi m^6},$$

and

$$\sigma \simeq \frac{25}{18}\times\frac{\lambda^4}{64\pi m^6}.$$

It is useful to slightly rewrite this expression. In order to do this, notice that a free non relativistic particle evolves with a dynamical phase factor e^{imt}, i.e. with frequency m; the associated wave-length is

$$\lambda_C = \frac{2\pi}{m},$$

which is called the Compton length. The Compton length corresponds heuristically with the spatial extension of the wave-function of a single particle.It is then natural to define the geometrical cross-section as:

$$\sigma_{geom} = \pi\lambda_C^2 = \frac{4\pi^3}{m^2}.$$

Then the physical cross section is written in terms of the geometrical one as:

$$\sigma = \frac{25}{18}\times\sigma_{geom}\times\left(\frac{\lambda}{4\pi m}\right)^4.$$

Besides an $O(1)$ numerical factor, this expression shows explicitly that the process is perturbative as long as

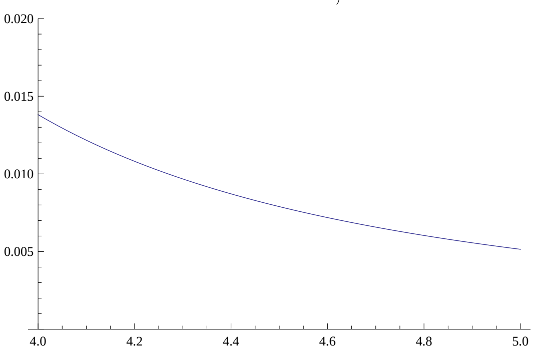
$$\lambda \ll 4\pi m.$$

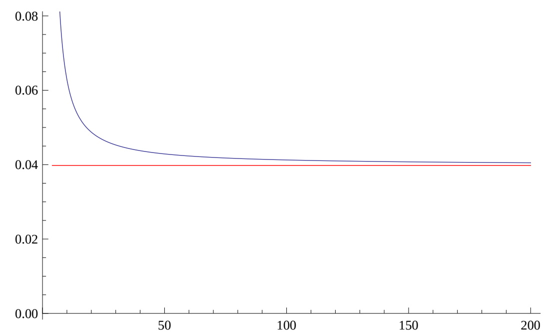
This is the correct regime of validity of pertubation theory, which is better by an $O(10)$ factor with respect to the naive expectation $\lambda \ll m$ from dimensional analysis. The appearence of 4π factors in perturbation theory is very common; for instance the perturbative regime of QED is $\alpha/4\pi \ll 1$ (where $\alpha = \frac{e^2}{4\pi}$).

For completeness, we give the result for the total cross section without approximations (from which one can deduce that the limits presented above are indeed correct), namely

$$\sigma = \frac{\lambda^4}{64\pi s}\left[\frac{4}{m^2\left(s-3m^2\right)} + \frac{2}{\left(m^2-s\right)^2} + \frac{4\log\left(\frac{m^2}{s-3m^2}\right)}{4m^4-5m^2 s+s^2} + \frac{4s\log\left(\frac{s}{m^2}-3\right)}{-8m^6+14m^4 s-7m^2 s^2+s^3}\right],$$

and we show its behavior as a function of s (in GeV^2) for $\lambda = m = 1$ GeV. The plot on the left displays 2σ in the limit of small masses, which correctly tends to $\frac{25}{576\pi}\text{GeV}^{-2}$ for this choice of parameters. The plot on the right shows the observable $2s^2 \times \sigma$ in the high energy region. The asymptotic flatness of the plot underlines the s^{-2} behavior of the cross section, and the value of the plotted quantity at $s \to \infty$ is $\frac{1}{8\pi}\text{GeV}^{-2}$.





You can refer to the Mathematica notebook to check this yourself.

Application

Suppose a particle with ultra-relativistic energy $E \gg m^2$ hits a box of targets of length ℓ. By definition of cross-section, the probability that a scattering event where the particle loses energy ΔE happens when the particle hits a target of thickness dx is given by:

$$\frac{dP}{d\Delta E} = \frac{d\sigma}{d\Delta E}\rho dx,$$

where ρ is the density of target particles. It follows that the mean energy loss per crossed length is:

$$\frac{d\langle \Delta E\rangle}{dx} = \int \frac{dP}{dx\, d\Delta E}\Delta E d\Delta E = \int \rho \frac{d\sigma}{d\Delta E}\Delta E d\Delta E.$$

To compute this in the relativistic regime, we just need to rewrite (1) in the Lab frame. In order to do this, notice that from the explicit expression of t, it follows:

$$\frac{d\sigma}{dt} = \frac{2}{s}\frac{d\sigma}{d\cos\theta^*},$$

where θ^* is the angle we considered so far in the center of mass frame. Once the cross section is written in terms of Mandelstam invariants, we can just evaluate them in the Lab frame, where momenta are parametrized as

$$\begin{aligned} p_a &= (E,0,0,p), \\ p_b &= (m,0,0,0), \\ p_c &= (E_c,0,p_c\sin\theta,p_c\cos\theta), \\ p_d &= (E_d,0,-p_d\sin\phi,p_d\cos\phi); \end{aligned}$$

$$\implies \quad t = (p_a - p_c)^2 = (p_d - p_b)^2 = 2m^2 - 2mE_d = 2m(E - E_c) = 2m\Delta E,$$

where we have used conservation of energy $E_a + m = E_c + E_d$. Thus we conclude

$$\langle \Delta E \rangle = \frac{\langle t \rangle}{2m}.$$

Supposing the target particles to be distinguishable from the incoming ones, we can neglect the s-channel and u-channel contributions in (1). Indeed we are assuming scattering with little energy difference and small angles $\theta \sim 0$, so one of the outgoing particles can be considered as the inital moving particle, being slightly deflected. As we discussed, the leading contribution in the $\theta \sim 0$ regime comes from the t-channel. Having distinguishable particles also means the phase space has to be multiplied by a factor of 2. As we argued before then, we can approximate the cross section with:

$$\frac{d\sigma}{dt} \approx \left[\frac{1}{t - m^2}\right]^2 \frac{\lambda^4}{16\pi s^2}.$$

As shown from the CM expression $-s + 2m^2 \leq t \leq 0$, then we find:

$$\begin{aligned}\frac{d\langle \Delta E \rangle}{dx} &= \frac{\rho}{2m}\int dt\, t \frac{d\sigma}{dt} \approx \frac{\rho}{2m} \times \frac{\lambda^4}{16\pi s^2}\int_{-s}^{0} dt\, t \left[\frac{1}{t - m^2}\right]^2 = \frac{\rho}{2m} \times \frac{\lambda^4}{16\pi s^2} \times \left[\frac{s}{s + m^2} - \log\frac{m^2 + s}{m^2}\right] \\ &\approx -\frac{\rho}{2m} \times \frac{\lambda^4}{16\pi s^2} \times \log\frac{s}{m^2}.\end{aligned}$$

In general to compute the total energy loss for a particle which crosses the whole box might be difficult, since due to scattering the particle can deviate from a straight trajectory. However our result on the cross-section suggests that scattering events at small angle dominate the process, so that we can suppose that the particle travels on a stright line. Indeed, the mean energy loss *per collision* is given by:

$$\frac{1}{\sigma_{tot}}\int dt\, \frac{t}{2m}\frac{d\sigma}{dt} \approx \frac{m}{2}\log\frac{s}{m^2} \ll E,$$

which confirms that events with low transferred energy dominate the process. We also suppose that ℓ is small enough so that we can neglect the dependence in s, i.e. in the energy on the particle, on the travelled distance x. We conclude that the total energy loss can be approximated as:

$$\langle \Delta E \rangle \approx -\ell \times \frac{\rho}{2m} \times \frac{\lambda^4}{16\pi s^2} \times \log\frac{s}{m^2}.$$

Exercise 2

Recalling the anticommutation relation of the Dirac matrices $\{\gamma^\mu\,, \gamma^\nu\} = 2\eta^{\mu\nu}$, and using the cyclic property of the trace, we have:

$$\begin{aligned}\mathrm{Tr}[\gamma^\mu\gamma^\nu] &= \mathrm{Tr}[\gamma^\nu\gamma^\mu] = \frac{1}{2}\mathrm{Tr}[\{\gamma^\mu,\, \gamma^\nu\}] = \eta^{\mu\nu}\mathrm{Tr}[1] = 4\eta^{\mu\nu}, \\ \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] &= 2\eta^{\mu\nu}\mathrm{Tr}[\gamma^\rho\gamma^\sigma] - \mathrm{Tr}[\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma] = 8\eta^{\mu\nu}\eta^{\rho\sigma} - \mathrm{Tr}[\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma] \\ &= 8\eta^{\mu\nu}\eta^{\rho\sigma} - 8\eta^{\mu\rho}\eta^{\nu\sigma} + \mathrm{Tr}[\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma] \\ &= 8\eta^{\mu\nu}\eta^{\rho\sigma} - 8\eta^{\mu\rho}\eta^{\nu\sigma} + 8\eta^{\mu\sigma}\eta^{\rho\nu} - \mathrm{Tr}[\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu].\end{aligned}$$

Using the cyclicity of the trace we have:

$$\mathrm{Tr}[\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu] = \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] \quad\implies\quad \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\rho\nu}).$$

Recalling the anticommutation relation $\{\gamma^\mu\,, \gamma^5\} = 0$ we can consider the trace of an odd number of γ matrices and insert $\gamma^5\gamma^5 = 1$:

$$\mathrm{Tr}[\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}] = \mathrm{Tr}[\gamma^5\gamma^5\,\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}].$$

Using the cyclicity we have

$$\mathrm{Tr}[\gamma^5\gamma^5\,\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}] = \mathrm{Tr}[\gamma^5\,\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}\gamma^5],$$

while if we were to pass γ_5 through all the other matrices we would get a minus sign for each anticommutation:

$$\mathrm{Tr}[\gamma^5\gamma^5\,\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}] = (-1)^{2n+1}\mathrm{Tr}[\gamma^5\,\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}\gamma^5],$$

so that

$$\mathrm{Tr}[\gamma^5\gamma^5\,\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}] = -\mathrm{Tr}[\gamma^5\gamma^5\,\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}] = 0.$$

From this results it follows straightforwardly that

$$\mathrm{Tr}[\gamma^5\cdot(\text{odd number of }\gamma\text{'s})] = \mathrm{Tr}[(\text{odd number of }\gamma\text{'s})] = 0,$$

since γ^5 can be written as the product of 4 Dirac matrices and hence the total number is still odd.

We now show that $\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] = 0$. Let's insert the factor $1 = \eta^{\rho\rho}\gamma^\rho\gamma^\rho$, (not summed over ρ), where $\rho \neq \mu$, $\rho \neq \nu$:

$$\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] = \eta^{\rho\rho}\mathrm{Tr}[\gamma^\rho\gamma^\rho\gamma^5\gamma^\mu\gamma^\nu] = -\eta^{\rho\rho}\mathrm{Tr}[\gamma^\rho\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho] = -\eta^{\rho\rho}\mathrm{Tr}[\gamma^\rho\gamma^\rho\gamma^5\gamma^\mu\gamma^\nu] = 0,$$

where we have anticommuted γ^ρ and used then cyclicity.

Finally let us consider

$$\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma].$$

First of all we notice that whenever two Lorentz indices are equal, this expression vanishes since it becomes proportional to $\mathrm{Tr}[\gamma^5\gamma\gamma]$. Moreover, because all indices are different, one gets a minus sign after every anticommutation of two Dirac matrices. Thus this object is completely antisymmetric in its Lorentz indices and cannot but be proportional to the Levi-Civita tensor:

$$\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = k\epsilon^{\mu\nu\rho\sigma}.$$

To work out the coefficient it is convenient to consider the particular case $\{\mu,\nu,\rho,\sigma\} = \{3,2,1,0\}$ and use the definition $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, to get

$$\mathrm{Tr}[i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3\gamma^2\gamma^1\gamma^0] = k\epsilon^{3210} \quad\Longrightarrow\quad -4i = k.$$

Thus

$$\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = -4i\epsilon^{\mu\nu\rho\sigma}.$$