Quantum Field Theory

Set 10: solutions

Exercise 1

Let us consider the Lagrangian of two interacting scalar fields:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}M^2\Phi^2 - \frac{\lambda}{2}\phi^2\Phi.$$

We want to study the decay $\Phi \to \phi\phi$. In what follows p_a will be the four-momentum of Φ while p_b and p_c are the four-momenta of the final particles. The differential decay width in the Φ center of mass reads

$$d\Gamma = \frac{1}{2M}|\mathcal{M}_{fi}|^2\sqrt{1-\frac{4m^2}{M^2}}\frac{d\cos\theta d\varphi}{32\pi^2},$$

where we have used the form of the 2 body phase space for equal final particles. We only need to compute the matrix element between the final state $|f\rangle$ and the initial state $|i\rangle$ which is defined by

$$S_{fi} \equiv \langle f|S|i\rangle = \delta_{fi} + i(2\pi)^4\delta^4(p_a - p_b - p_c)\mathcal{M}_{fi}.$$

Let's first obtain the expression for the S-matrix element from the operator equation

$$S = T\exp\left(-i\int_{-\infty}^{\infty} dt\ H_I(t)\right) \equiv \mathbb{1} - i\int_{-\infty}^{\infty} dt\ H_I(t) + (-i)^2\int_{-\infty}^{\infty} dt\ H_I(t)\int_{-\infty}^{t} dt'\ H_I(t') + ...,$$

where T is the time-ordering symbol (and it is defined by the previous Taylor expansion), and H_I is the interaction Hamiltonian written in the interaction picture, $H_I(t) = e^{iH_0t}H_{int}(0)e^{-iH_0t}$. In the Born approximation, one only retains the first two terms in the Taylor expansion, so that

$$\begin{aligned}\langle f|S|i\rangle &\simeq \langle f|\left(\mathbb{1} - i\int_{-\infty}^{\infty} dt\ H_I(t)\right)|i\rangle = \delta_{fi} - i\int_{-\infty}^{\infty} dt\ \langle f|e^{iH_0t}H_{int}(0)e^{-iH_0t}|i\rangle \\ &= \delta_{fi} - i\int_{-\infty}^{\infty} dt\ \langle f|e^{iE_ft}H_{int}(0)e^{-iE_it}|i\rangle = \delta_{fi} - i\langle f|H_{int}(0)|i\rangle\int_{-\infty}^{\infty} dt\ e^{i(E_f-E_i)t} \\ &= \delta_{fi} - 2\pi i\delta(E_f - E_i)\langle f|H_{int}(0)|i\rangle.\end{aligned}$$

Thus we can identify the amplitude:

$$(2\pi)^3\delta^3(\vec{p}_f - \vec{p}_i)\mathcal{M}_{fi} = -\langle f|H_{int}(0)|i\rangle.$$

Let us now use the free decomposition for real fields in order to compute the matrix element of the interaction hamiltonian.

$$\Phi(\vec{x},0) = \int d\Omega_{\vec{p}}\left(a(\vec{p})e^{i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right),$$
$$\phi(\vec{x},0) = \int d\Omega_{\vec{p}}\left(b(\vec{p})e^{i\vec{p}\cdot\vec{x}} + b^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right).$$

Therefore in the rest frame of Φ ($\vec{p}_a = \vec{0}$) we have

$$|i\rangle = |\vec{p}_a\rangle = |\vec{0}\rangle = a^\dagger(\vec{0})|0\rangle, \qquad |f\rangle = |\vec{p}_b,\vec{p}_c\rangle = b^\dagger(\vec{p}_b)b^\dagger(\vec{p}_c)|0\rangle,$$

and the amplitude is

$$\langle\vec{p}_b,\vec{p}_c|H_{int}(0)|\vec{p}_a\rangle = \langle 0|b(\vec{p}_b)b(\vec{p}_c)H_{int}(0)a^\dagger(\vec{0})|0\rangle = \frac{\lambda}{2}\int d^3x\ \langle 0|b(\vec{p}_b)b(\vec{p}_c)\ \Phi(\vec{x},0)\phi^2(\vec{x},0)\ a^\dagger(\vec{0})|0\rangle =$$
$$\frac{\lambda}{2}\int d^3x\ d\Omega_{\vec{q}}d\Omega_{\vec{k}}d\Omega_{\vec{t}}\ \langle 0|b(\vec{p}_b)b(\vec{p}_c)b^\dagger(\vec{t})b^\dagger(\vec{k})a(\vec{q})a^\dagger(\vec{0})|0\rangle e^{-i(\vec{q}-\vec{k}-\vec{t})\cdot\vec{x}},$$

where we have taken only the proper terms in order to have the same number of a, $a^\dagger$ and of b, $b^\dagger$ (only a state with the same number of daggered and undaggered operators has non zero overlap with $|0\rangle$). Making use of the equal time commutation relations

$$[a(\vec{p}), a^\dagger(\vec{q})] = [b(\vec{p}), b^\dagger(\vec{q})] = (2\pi)^3 2p_0 \delta^3(\vec{p}-\vec{q}),$$

we get

$$\langle \vec{p}_b, \vec{p}_c | H_{int}(0) | \vec{0}\rangle = \frac{\lambda}{2}\int d^3x\, d\Omega_{\vec{q}} d\Omega_{\vec{k}} d\Omega_{\vec{t}}\, e^{-i(\vec{q}-\vec{k}-\vec{t})\cdot\vec{x}} (2\pi)^9 2q_0 2k_0 2t_0 \delta^3(\vec{q}) \left[\delta^3(\vec{k}-\vec{p}_b)\delta^3(\vec{t}-\vec{p}_c) + \delta^3(\vec{k}-\vec{p}_c)\delta^3(\vec{t}-\vec{p}_b)\right].$$

Performing all the integrations on momenta and recalling that $\int d^3x\; e^{-i(\vec{q}-\vec{k}-\vec{t})\cdot\vec{x}} = (2\pi)^3\delta^3(\vec{q}-\vec{k}-\vec{t})$ we finally get

$$\langle \vec{0} | H_{int}(0) | \vec{p}_b, \vec{p}_c\rangle = \lambda (2\pi)^3 \delta^3(\vec{p}_c + \vec{p}_b) = \langle \vec{p}_b, \vec{p}_c | H_{int}(0) | \vec{0}\rangle,$$

where the last equality holds since λ is real. Therefore

$$(2\pi)^3\delta^3(\vec{p}_b+\vec{p}_c)\mathcal{M}_{fi} = -\lambda(2\pi)^3\delta^3(\vec{p}_b+\vec{p}_c) \quad\Longrightarrow\quad \mathcal{M}_{fi} = -\lambda.$$

Finally the decay width reads

$$\frac{d\Gamma}{d\cos\theta} = \frac{\lambda^2}{32\pi M}\sqrt{1-\frac{4m^2}{M^2}},$$

and in the end

$$\Gamma = \frac{\lambda^2}{32\pi M}\sqrt{1-\frac{4m^2}{M^2}}.$$

In the last formula we have integrated over half of the phase space since the two final-state particles are identical. Notice that the decay width goes to zero when $m \to M/2$ which is correct because for bigger values of the final state mass the decay is kinematically forbidden. The lifetime τ is simply $1/\Gamma$.

Exercise 2

This exercise could be solved along the lines of the previous, but since the interaction Hamiltonian has four identical fields, and the scattering is $2 \to 2$, then the blind application of the above formulas rapidly becomes unfeasible. Then we can use Wick's theorem to evaluate the matrix element efficiently.
Recall that Wick's theorem states

$$T\{\phi(x_1)\cdots\phi(x_n)\} \equiv\, :[\phi(x_1)\cdots\phi(x_n) + \text{contractions}]: \;,$$

where ": :" is the normal-ordering symbol, meaning that all the creation operators appear on the left, while "contractions" is a sketchy way to indicate all the possible contractions of two fields, namely

$$\wick{\c\phi(x_1)\c\phi(x_2)} = D(x_1-x_2),$$

D representing the propagator. To be concrete, let's work out the T-ordered product of four identical scalar fields (understood in the interaction picture), according to Wick's theorem:

$$\begin{aligned}
T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\} \equiv\; & :[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \\
& + \wick{\c\phi(x_1)\c\phi(x_2)\phi(x_3)\phi(x_4)} + \wick{\c\phi(x_1)\phi(x_2)\c\phi(x_3)\phi(x_4)} + \wick{\c\phi(x_1)\phi(x_2)\phi(x_3)\c\phi(x_4)} \\
& + \wick{\phi(x_1)\c\phi(x_2)\c\phi(x_3)\phi(x_4)} + \wick{\phi(x_1)\c\phi(x_2)\phi(x_3)\c\phi(x_4)} + \wick{\phi(x_1)\phi(x_2)\c\phi(x_3)\c\phi(x_4)} \\
& + \wick{\c1\phi(x_1)\c1\phi(x_2)\c2\phi(x_3)\c2\phi(x_4)} + \wick{\c1\phi(x_1)\c2\phi(x_2)\c1\phi(x_3)\c2\phi(x_4)} + \wick{\c1\phi(x_1)\c2\phi(x_2)\c2\phi(x_3)\c1\phi(x_4)}]: \\
=\; & :\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4): \\
& + D(x_1-x_2):\phi(x_3)\phi(x_4): + D(x_1-x_3):\phi(x_2)\phi(x_4): + D(x_1-x_4):\phi(x_2)\phi(x_3): \\
& + D(x_2-x_3):\phi(x_1)\phi(x_4): + D(x_2-x_4):\phi(x_1)\phi(x_3): + D(x_3-x_4):\phi(x_1)\phi(x_2): \\
& + D(x_1-x_2)D(x_3-x_4) + D(x_1-x_3)D(x_2-x_4) + D(x_1-x_4)D(x_2-x_3). \qquad (1)
\end{aligned}$$

To compute the scattering amplitude $\mathcal{M}_{fi}$ we need to evaluate

$$\langle \vec{p}_c, \vec{p}_d | T \left(-i \int_{-\infty}^{\infty} dt \; H_I(t) \right) | \vec{p}_a, \vec{p}_b \rangle = -\frac{i\lambda}{4!} \langle \vec{p}_c, \vec{p}_d | T \int d^4x \; \phi^4(x) | \vec{p}_a, \vec{p}_b \rangle,$$

Note that $T(\phi(x)^4) = \phi(x)^4$, since the four fields are evaluated at the same time x^0. Nonetheless, it turns out convenient to write the ordinary product as a time-ordered product, because the application of Wick's theorem on $T(\phi(x)^4)$ gives huge simplifications. We can thus use the explicit form found in equation (1).

The last line in (1) only contains C-numbers (pairs of propagators), so the contribution coming from there is non vanishing only if initial and final state momenta are identical. This corresponds to the trivial part of the S-matrix, so we discard it because, computing scattering amplitudes, we always look for contribution coming from nontrivial kinematics.
If we now consider the terms with one propagator, the structure we find is

$$A \equiv \langle 0 | a(\vec{p}_c) a(\vec{p}_d) a^\dagger(\vec{k}) a(\vec{q}) a^\dagger(\vec{p}_a) a^\dagger(\vec{p}_b) | 0 \rangle,$$

where the terms with 2 a's and 0 $a^\dagger$'s or viceversa have not been written because vanishing. Moving the creation operator to the left and the annihilation operator to the right one gets

$$\begin{aligned} A \propto \; & \delta^3(\vec{p}_c - \vec{p}_b)\delta^3(\vec{p}_d - \vec{k})\delta^3(\vec{p}_a - \vec{q}) + \delta^3(\vec{p}_c - \vec{p}_a)\delta^3(\vec{q} - \vec{p}_b)\delta^3(\vec{p}_d - \vec{k}) \\ & + \delta^3(\vec{p}_c - \vec{p}_k)\delta^3(\vec{q} - \vec{p}_a)\delta^3(\vec{p}_d - \vec{p}_b) + \delta^3(\vec{q} - \vec{p}_b)\delta^3(\vec{p}_c - \vec{k})\delta^3(\vec{p}_d - \vec{p}_a), \end{aligned}$$

in which always appears a Dirac delta involving two external momenta. So again, these pieces contribute to the trivial part of the scattering amplitude (the nontrivial one being the one in which the incoming and outgoing momenta are not related to each other).
The only contribution comes from the term with four uncontracted fields.
Let's start by defining $\phi^+(x)$ ($\phi^-(x)$) as the component of the field associated with the creation (annihilation) operator, and noticing that

$$\begin{aligned} \phi^-(x) | \vec{p} \rangle &= \int d\Omega_{\vec{k}} \; e^{-ikx} a(\vec{k}) a^\dagger(\vec{p}) | 0 \rangle = e^{-ipx} | 0 \rangle, \\ \langle \vec{p} | \phi^+(x) &= \int d\Omega_{\vec{k}} \; e^{ikx} \langle 0 | a(\vec{p}) a^\dagger(\vec{k}) = \langle 0 | e^{ipx}, \end{aligned}$$

where we have used the equal-time commutation relations.
Now, in terms of these components at defined frequency, one gets

$$\langle \vec{p}_c, \vec{p}_d | : \phi(x)^4 : | \vec{p}_a, \vec{p}_b \rangle = 6 \; \langle \vec{p}_c, \vec{p}_d | \phi^+(x)^2 \phi^-(x)^2 | \vec{p}_a, \vec{p}_b \rangle = 6 \cdot 2 \cdot 2 \; e^{i(p_c + p_d - p_a - p_b)x},$$

where we have retained only terms with 2 creators and 2 annihilators. The two factors of 2 come from commuting $\phi^-(x)^2$ at the right of $a^\dagger(\vec{p}_a) a^\dagger(\vec{p}_b)$ and $\phi^+(x)^2$ at the left of $a(\vec{p}_c) a(\vec{p}_d)$. In fact one has

$$\begin{aligned} \int d\Omega_{\vec{k}} d\Omega_{\vec{q}} a(\vec{q}) a(\vec{k}) a^\dagger(\vec{p}_a) a^\dagger(\vec{p}_b) &= \int d\Omega_{\vec{k}} d\Omega_{\vec{q}} \left[(2\pi)^3 2k_0 \delta^3(\vec{p}_a - \vec{k}) a(\vec{q}) a^\dagger(\vec{p}_b) + a(\vec{q}) a^\dagger(\vec{p}_a) a(\vec{k}) a^\dagger(\vec{p}_b) \right] \\ \int d\Omega_{\vec{k}} d\Omega_{\vec{q}} & \left[(2\pi)^3 2k_0 (2\pi)^3 2q_0 \delta^3(\vec{p}_a - \vec{k}) \delta^3(\vec{p}_b - \vec{q}) + (2\pi)^3 2k_0 (2\pi)^3 2q_0 \delta^3(\vec{p}_a - \vec{q}) \delta^3(\vec{p}_b - \vec{k}) \right] = 2. \end{aligned}$$

From what we have obtained here, we can deduce a set of rules to evaluate matrix elements of time-ordered products. First of all, to obtain scattering amplitudes, one can discard all terms that lead to a Dirac delta between two external momenta, since these will contribute to the trivial part of the S-matrix. Second, once everything is normal-ordered, the amplitude will receive a contribution from every commutation of a ϕ^- (ϕ^+) trough a creation (annihilation) operator.
We can better reformulate these two rules by defining

$$\begin{aligned} \overbrace{\phi(x) | \vec{p}}\rangle \equiv \phi^-(x) | \vec{p} \rangle &= e^{-ipx} | 0 \rangle, \\ \langle \overbrace{\vec{p} | \phi}(x) \equiv \langle \vec{p} | \phi^+(x) &= \langle 0 | e^{ipx}, \end{aligned}$$

as the contractions of fields with external states, and by stating that

- a scattering amplitude gets a vanishing contribution when *not* all external states are contracted with fields (trivial part of the S-matrix),
- a scattering amplitude (for scalars) gets a contribution equal to e^{-ipx} (e^{ipx}) for each contraction of an initial (final) state with momentum $\vec{p}$ with a field $\phi(x)$.

The latter item is the Feynman rule for external legs in coordinate space. Next time a complete set of rules (including also internal propagators) will be provided in momentum space.

At the end of the day, one gets:

$$-\frac{i\lambda}{4!}\langle\vec{p}_c,\vec{p}_d|T\int d^4x\ \phi^4(x)|\vec{p}_a,\vec{p}_b\rangle = -\frac{i\lambda}{4!}\int d^4x\ (4!)\ e^{i(p_c+p_d-p_a-p_b)x} = -i\lambda(2\pi)^4\delta^4(p_c+p_d-p_a-p_b)$$
$$= i\mathcal{M}_{fi}(2\pi)^4\delta^4(p_c+p_d-p_a-p_b),$$

thus again $\mathcal{M}_{fi} = -\lambda$.
From this expression it is now straightforward to deduce the total cross section which, using the standard definitions, is

$$d\sigma = \frac{1}{2s}\lambda^2\frac{d\Omega}{32\pi^2} \quad\Longrightarrow\quad \sigma = \frac{\lambda^2}{32\pi s}.$$

In the last formula we have integrated on half of the phase space since the two final-state particles are identical.

Exercise 3 (optional)

In many problems in quantum mechanics, including scattering theory, the Hamiltonian of the system can be splitted as,

$$H = H_0 + H_I,$$

where H_0 corresponds to a free non interacting system and H_I encodes the interactions. In many applications it is useful to work in a representation where only the interactions contribute to evolution of states, neglecting the trivial action of free evolution. Then, given a state $|\psi(t)\rangle_S$ and an operator $\mathcal{O}_S$ in Schrodinger picture, we define interaction picture states and operators as

$$|\psi(t)\rangle_I = e^{iH_0t}\,|\psi(t)\rangle_S\,, \qquad\qquad \mathcal{O}_I(t) = e^{iH_0t}\mathcal{O}_S e^{-iH_0t}.$$

The interaction picture is thus in between the Schrodinger picture, where states evolve in time, and the Heisenberg picture, where they are time independent. The evolution operator in the interaction picture is found as follows:

$$|\psi(t)\rangle_I = e^{iH_0t}\,|\psi(t)\rangle_S = e^{iH_0t}e^{-iH(t-t')}\,|\psi(t')\rangle_S = \underbrace{e^{iH_0t}e^{-iH(t-t')}e^{-iH_0t'}}_{=U(t,t')}|\psi(t')\rangle_S \equiv U(t,t')\,|\psi(t')\rangle_I\,.$$

The usual properties are seen to hold:

$$U(t,t')U(t',t'') = U(t,t''), \qquad\qquad U^\dagger(t,t') = U^{-1}(t,t') = U(t',t). \tag{2}$$

From its explicit expression we deduce:

$$i\frac{\partial U(t,t')}{\partial t} = -H_0U(t,t') + e^{iH_0t}He^{-iH(t-t')}e^{-iH_0t'} = \underbrace{e^{iH_0t}(H-H_0)e^{-iH_0t}}_{=H_I(t)}U(t,t') \equiv H_I(t)U(t,t'), \tag{3}$$

where we defined the interaction picture Hamiltonian $H_I(t)$. Eq. (3) must be solved with the obvious initial condition $U(t',t') = \mathbb{1}$. Then it is easy to verify that the solution is

$$U(t,t') = \mathbb{1} - i\int_{t'}^{t} d\tau H_I(\tau)U(\tau,t'). \tag{4}$$

Indeed it obviously satisfies $i\frac{\partial U(t,t')}{\partial t} = H_I(t)U(t,t')$. Furthermore for $t = t'$ the second term vanishes, giving $U(t',t') = \mathbb{1}$. This proves that (4) is solution of (3) satisfying the initial condition. Notice that from (4) the Dyson series follows immediately:

$$U(t,t') = \mathbb{1} - i\int_{t'}^{t} dt_1 H_I(t_1)U(t_1,t') = \mathbb{1} - i\int_{t'}^{t} dt_1 H_I(t_1)\left(\mathbb{1} - i\int_{t'}^{t_1} dt_2 H_I(t_2)U(t_2,t')\right)$$
$$= \mathbb{1} - i\int_{t'}^{t} dt_1 H_I(t_1) - i^2\int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 H_I(t_1)H_I(t_2) + \ldots = Te^{-i\int_{t'}^{t} d\tau H_I(\tau)}.$$

Now let's recall the definition of *in* (*out*) states and the associated Moeller operator. Given a free state $|\phi_\alpha\rangle$, an *in* (*out*) state $|\psi_\alpha^+\rangle$ ($|\psi_\alpha^-\rangle$) is one such that for $t \to -\infty$ ($t \to \infty$) is well desribed by it:

$$|\psi_\alpha^\pm(t)\rangle = e^{-iHt}|\psi_\alpha^+\rangle \overset{t\to\mp\infty}{\longrightarrow} |\phi_\alpha(t)\rangle = e^{-iH_0 t}|\phi_\alpha\rangle\,,$$
$$\Longrightarrow \quad |\psi_\alpha^\pm\rangle = \lim_{t\to\mp\infty} e^{iHt}e^{-iH_0 t}|\phi_\alpha\rangle \equiv \Omega_\pm|\phi_\alpha\rangle\,;$$

We see immediately that

$$U(0,-\infty) = \lim_{t\to-\infty} U(0,t) = \lim_{t\to-\infty} e^{iHt}e^{-iH_0 t} = \Omega_+, \tag{5}$$

and

$$U(\infty,0) = \lim_{t\to\infty} U(t,0) = \lim_{t\to+\infty} e^{iH_0 t}e^{-iHt} = \Omega_-^\dagger.$$

which imply

$$\Longrightarrow \quad S = U(\infty,-\infty) = U(\infty,0)U(0,-\infty) = \Omega_-^\dagger\Omega_+.$$

Similarly it follows:

$$U(t,-\infty) = \lim_{t'\to-\infty} e^{iH_0 t}e^{-iH(t-t')}e^{-iH_0 t'} = e^{iH_0 t}\left(\lim_{t'\to-\infty} e^{iH(t'-t)}e^{-iH_0(t'-t)}\right)e^{-iH_0 t} = e^{iH_0 t}\Omega_+ e^{-iH_0 t}. \tag{6}$$

Here we used the fact that the definition of Ω_+ does not depend on the choice of the reference time. Similarly you can show

$$U(\infty,t) = e^{iH_0 t}\Omega_-^\dagger e^{-iH_0 t}.$$

Finally we are in position to prove that the Lippman-Schwinger equation follows from eq. (4). Indeed from (4), (5) and (6) it follows:

$$\Omega_+ = \mathbb{1} - i\int_{-\infty}^{0} dt\, H_I(t)e^{iH_0 t}\Omega_+ e^{-iH_0 t}.$$

Applying now this on a an eigenstate $|\phi_\alpha\rangle$ of the free Hamiltonian H_0, we get

$$\overbrace{\Omega_+|\phi_\alpha\rangle}^{=|\psi_\alpha^+\rangle} = |\phi_\alpha\rangle - i\int_{-\infty}^{0} dt\, H_I(t)e^{iH_0 t}\Omega_+ e^{-iH_0 t}|\phi_\alpha\rangle$$
$$= |\phi_\alpha\rangle - i\int_{-\infty}^{0} dt\, H_I(t)e^{i(H_0-E_\alpha)t}\Omega_+|\phi_\alpha\rangle$$
$$= |\phi_\alpha\rangle - i\int_{-\infty}^{0} dt\, e^{iH_0 t}(H-H_0)e^{-iH_0 t}e^{i(H_0-E_\alpha)t}|\psi_\alpha^+\rangle$$
$$= |\phi_\alpha\rangle - i\int_{-\infty}^{0} dt\, e^{i(H_0-E_\alpha)t}(H-H_0)|\psi_\alpha^+\rangle\,.$$

To evaluate the integral we must shift $E_\alpha \to E_\alpha + i\varepsilon$ to make it convergent. Then, inserting a complete set of eigenstates of H_0, we get

$$\int_{-\infty}^{0} dt\, e^{i(H_0-E_\alpha-i\varepsilon)t}\sum_n |\psi_n\rangle\langle\psi_n| = \sum_n \int_{-\infty}^{0} dt\, e^{i(H_0-E_\alpha-i\varepsilon)t}|\psi_n\rangle\langle\psi_n| = \sum_n \int_{-\infty}^{0} dt\, e^{i(E_n-E_\alpha-i\varepsilon)t}|\psi_n\rangle\langle\psi_n|$$
$$= \sum_n \frac{-i}{E_n - E_\alpha - i\varepsilon}|\psi_n\rangle\langle\psi_n| = \frac{i}{E_\alpha - H_0 + i\varepsilon}.$$

Substituiting back we obtain Lippman-Schwinger equation for *in* states:

$$|\psi_\alpha^+\rangle = |\phi_\alpha\rangle + \frac{1}{E_\alpha - H_0 + i\varepsilon}(H - H_0)\,|\psi_\alpha^+\rangle\,.$$

To prove the Lippman-Schwinger equation for *out* states, notice that $U(0,\infty) = \Omega_-$. Then an analogous derivation shows that:

$$|\psi_\alpha^-\rangle = |\phi_\alpha\rangle + \frac{1}{E_\alpha - H_0 - i\varepsilon}(H - H_0)\,|\psi_\alpha^-\rangle\,.$$