Quantum Field Theory

Set 9

Exercise 1: Decay amplitudes in the center of mass

Consider the decay $A \to C\,D$. The decay width is given by

$$d\Gamma_{A\to CD} = \frac{1}{2E_A} \left|\mathcal{M}_{A\to CD}\right|^2 d\Phi_2 \,.$$

Compute the above expression in the rest frame of A.
Consider then the 3-body decay $A \to B\,C\,D$. The only difference here is the presence of the 3-body phase space $d\Phi_3$. Let us consider the case where we only deal with scalar particles in the initial and final state or we sum and average over all possible polarization of non scalar particles. Hence the Lorentz invariant matrix element $\left|\mathcal{M}_{A\to BCD}\right|^2$ can only depend on the three invariant s, t, u, where this time

$$s = (P_A - P_B)^2 = (P_C + P_D)^2, \qquad t = (P_A - P_C)^2 = (P_B + P_D)^2, \qquad u = (P_A - P_D)^2 = (P_B + P_C)^2.$$

Notice that under these assumptions the system is invariant under rotations. Therefore we can freely integrate over three angular variables since $\left|\mathcal{M}_{A\to BCD}\right|^2$ won't depend on them. Performing all the integrations required, show that in the rest frame of A we have:

$$d\Gamma_{A\to BCD} = \frac{1}{2M} \left|\mathcal{M}_{A\to BCD}\right|^2 \frac{dE_B\,dE_C}{4(2\pi)^3},$$

where $M^2 = P_A^2$. Finally express the differential decay amplitude in terms of s, t: $\frac{d\Gamma}{ds\,dt}$.

Exercise 2: Free scalar propagator

Consider the scalar free propagator

$$\mathcal{D}(x-y) = \langle 0|\, T(\phi(x)\phi(y))\, |0\rangle \ ,$$

where the time ordered product is defined by

$$T(\phi(x)\phi(y)) \equiv \theta(x_0 - y_0)\phi(x)\phi(y) + \theta(y_0 - x_0)\phi(y)\phi(x) \ .$$

Making use of the Klein-Gordon equation for the field ϕ show that:

$$\left(\Box_x + m^2\right)\mathcal{D}(x-y) = \frac{\partial}{\partial x^0}\delta(x_0 - y_0)\, \langle 0|\, [\phi(x)\,,\,\phi(y)]\, |0\rangle + 2\delta(x_0 - y_0)\, \langle 0|\, [\partial_0\phi(x)\,,\,\phi(y)]\, |0\rangle \,.$$

Consider the above expression as a distribution that makes sense only when inserted in an integral. Hence "integrate by parts" the $\delta^{'}(x_0 - y_0)$ and set $x_0 = y_0$ everywhere. Finally, make use of the canonical commutation relations to show that

$$\left(\Box_x + m^2\right)\mathcal{D}(x-y) = -i\delta^4(x-y).$$

One can solve the above equation in momentum space. Show that the solution is

$$\mathcal{D}(x-y)_\pm = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 \pm i\varepsilon} e^{-ip(x-y)}. \tag{1}$$

The above uncertainty in the sign of $i\varepsilon$ comes from the fact that there are two ways to regularize the integral in the complex plane in a Lorentz invariant way. We need to choose the right one. In order to do this, expand the field ϕ in $a\,a^\dagger$ and show that

$$\mathcal{D}(x-y) = \theta(x_0 - y_0)\int d\Omega_p e^{-ip(x-y)} + \theta(y_0 - x_0)\int d\Omega_p e^{+ip(x-y)}.$$

Finally show that only $\mathcal{D}(x-y)_+$ reproduces the above result: compute $\mathcal{D}(x-y)_+$ from equation (1) integrating over p_0 separately in the two cases $(x_0 - y_0) > 0$ and $(x_0 - y_0) < 0$ and closing the contour in the complex plane appropriately.

Exercise 3: Free fermion propagator

Consider the free propagator of a fermion

$$\mathcal{S}(x-y) = \langle 0| \, T(\psi(x)\bar{\psi}(y)) \, |0\rangle \, ,$$

where the time ordered product is defined by

$$T(\psi(x)\bar{\psi}(y)) \equiv \theta(x_0-y_0)\psi(x)\bar{\psi}(y) - \theta(y_0-x_0)\bar{\psi}(y)\psi(x) \ .$$

Making use of the Dirac equation for the field ψ show that:

$$(i\,\partial\!\!\!/_x - m)\,\mathcal{S}(x-y) = i\delta^4(x-y).$$

One can solve the above equation in momentum space. Show that the solution is

$$\mathcal{S}(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\tilde{S}(p)e^{-ip(x-y)}, \qquad \text{with} \qquad \tilde{S}(p) = \frac{i}{p\!\!\!/ - m} \equiv i\frac{p\!\!\!/ + m}{p^2 - m^2 + i\varepsilon}.$$

Exercise 4: Free vector propagator

Consider the Lagrangian of a massless vector field:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\left(1 - \frac{1}{\alpha}\right)(\partial_\mu A^\mu)^2.$$

Find the equation of motion for the field A_μ. Consider the free vector propagator:

$$\mathcal{D}_{\rho\sigma}(x-y) = \langle 0| \, T(A_\rho(x)A_\sigma(y)) \, |0\rangle \, ,$$

where the time ordered product is defined by

$$T(A_\rho(x)A_\sigma(y)) \equiv \theta(x_0-y_0)A_\rho(x)A_\sigma(y) + \theta(y_0-x_0)A_\sigma(y)A_\rho(x) \ .$$

Making use of the equation for the field A show that:

$$\left(\Box\delta^\mu_\nu - \left(1 - \frac{1}{\alpha}\right)\partial^\mu\partial_\nu\right)\mathcal{D}_{\mu\sigma}(x-y) = i\delta^4(x-y)\eta_{\nu\sigma}.$$

One can solve the above equation in momentum space. Show that the solution is

$$\mathcal{D}_{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\tilde{D}_{\mu\nu}(p)e^{-ip(x-y)}, \qquad \text{with} \qquad \tilde{D}_{\mu\nu}(p) = \frac{-i}{p^2 + i\varepsilon}\left(\eta_{\mu\nu} - (1-\alpha)\frac{p_\mu p_\nu}{p^2}\right).$$