Quantum Field Theory

Set 7

Exercise 1: Charge conjugation of Dirac Lagrangian

Recalling the transformation properties of Weyl fermions under charge conjugation

$$C^\dagger \chi_L C = \eta_L \epsilon \chi_R^*,$$
$$C^\dagger \chi_R C = \eta_R \epsilon \chi_L^*,$$

show that the Dirac action

$$S = \int d^4x \, \left(i\chi_L^\dagger \bar{\sigma}^\mu \partial_\mu \chi_L + i\chi_R^\dagger \sigma^\mu \partial_\mu \chi_R - m(\chi_R^\dagger \chi_L + \chi_L^\dagger \chi_R) \right)$$

is invariant only for the choice $\eta_R^* \eta_L = -1$. Derive the matrix U_C that describes the transformation properties of a Dirac fermion according to the following formula:

$$C^\dagger \psi C = \eta_C U_C \bar{\psi}^T, \qquad \psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}.$$

Show that $U_C = i\gamma^0\gamma^2$.

Exercise 2: Charge conjugation properties of a particle-antiparticle system

Consider a scalar particle-antiparticle pair in the center of mass frame. Assume that their total angular momentum is l. Hence this state can be written as

$$|\Phi_l\rangle = \int d\Omega_{\vec{p}} \; f_l(\vec{p}, -\vec{p}) a^\dagger(\vec{p}) b^\dagger(-\vec{p}) |0\rangle.$$

Recalling the symmetry properties of a state with angular momentum l, namely $f_l(\vec{p}, -\vec{p}) = (-1)^l f_l(-\vec{p}, \vec{p})$, and the action of charge conjugation on scalars

$$C a^\dagger(\vec{k}) C = \eta_C^* b^\dagger(\vec{k}), \qquad C b^\dagger(\vec{k}) C = \eta_C a^\dagger(\vec{k}),$$

find the transformation properties of the state $|\Phi_l\rangle$ under C.
Consider now a generic state composed of a fermionic particle-antiparticle pair with angular momentum l and total spin S:

$$|\Psi_{l,S}\rangle = \sum_{r,t=1}^{2} \int d\Omega_{\vec{p}}, f_l(\vec{p}, -\vec{p}) \, \chi_S(r,t) \, b^\dagger(r, \vec{p}) \tilde{d}^\dagger(t, -\vec{p}) |0\rangle,$$

where $\tilde{d}^\dagger(t, -\vec{p}) \equiv d^\dagger(t', -\vec{p}) \epsilon^{tt'}$. The action of charge conjugation is defined as

$$C b^\dagger(r, \vec{k}) C = -\eta_C^* \tilde{d}^\dagger(r, \vec{k}), \qquad C d^\dagger(r, \vec{k}) C = \eta_C \tilde{b}^\dagger(r, \vec{k}),$$

and the wave functions satisfy

$$f_l(\vec{p}, -\vec{p}) = (-1)^l f_l(-\vec{p}, \vec{p}), \qquad \chi_S(r,t) = (-1)^{S+1} \chi_S(t, r).$$

Find the transformation properties of the state $|\Psi_{l,S}\rangle$ under C.

Exercise 3: Transformation properties of fermionic bilinears

Knowing the transformation properties of a Dirac fermion ψ under charge conjugation,

$$C^\dagger \psi(t,\vec{x}) C = -i\eta_C \gamma^2 \psi^*(t,\vec{x}),$$

deduce the transformation properties of all the bilinears of the form $\bar{\psi}\Gamma\psi$, where Γ is an element of the usual basis

$$\Gamma = \{1_4, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \gamma^{\mu\nu}\}.$$

Exercise 4: local interactions and superposition principle

Consider a free Klein Gordon theory $\varphi(x)$. Given an operator $\mathcal{O}$, define its *normal ordered* form, denoted by $:\mathcal{O}:$, as the operator obtained writing by hand all creation operators to the left of all destruction operators. Thus for instance $: a_p a_p^\dagger := a_p^\dagger a_p$. Similarly, writing $\varphi(x)$ as a sum of the positive and the negative frequency part:

$$\varphi(x) = \varphi_+(x) + \varphi_-(x),$$

$$\varphi_+(x) = \int d\Omega_k\, e^{ikx} a_k^\dagger, \qquad \varphi_-(x) = \int d\Omega_k\, e^{-ikx} a_k,$$

we get:

$$: \varphi(x)^2 :\equiv \varphi_+^2 + \varphi_-^2 + 2\varphi_+\varphi_-,$$
$$: \varphi(x)^4 :\equiv \varphi_+^4 + 4\varphi_+^3\varphi_- + 6\varphi_+^2\varphi_-^2 + 4\varphi_+\varphi_-^3 + \varphi_-^4.$$

- Consider an orthonormalized one particle state $|\Psi\rangle = \int d\Omega_k \hat{f}(k) a_k^\dagger |0\rangle$ and define $f(x) \equiv \int d\Omega_k \hat{f}(k) e^{-ikx}$. Compute the expectation value $\langle\Psi| : \varphi(x)^2 : |\Psi\rangle$
- Consider now an orthonormalized two particle state $|\Psi\rangle = \int d\Omega_1 d\Omega_2 \hat{f}_1(k_1)\hat{f}_2(k_2) a_{k_1}^\dagger a_{k_2}^\dagger |0\rangle$. Work in the asymptotic limit where f_1 and f_2 are spacially separated (for instance $f_1(k) = f_2(k) e^{i\vec{R}\cdot\vec{k}}$ with $|\vec{R}| \to \infty$). Compute $\langle\Psi|\Psi\rangle$, $\langle\Psi| : \varphi(x)^2 : |\Psi\rangle$ and $\langle\Psi| : \varphi(x)^4 : |\Psi\rangle$. Discuss the result.
- Now consider two wave-packets with width Δ centered in $\mathbf{k}_0$ and $-\mathbf{k}_0$ in momentum space meaning:

$$f_1(\mathbf{k}) = \frac{2\sqrt{2}\pi^{3/4}}{\Delta^{3/2}} e^{-\frac{|\mathbf{k}-\mathbf{k}_0|^2}{2\Delta^2} + i\mathbf{k}\mathbf{a}},$$
$$f_2(\mathbf{k}) = \frac{2\sqrt{2}\pi^{3/4}}{\Delta^{3/2}} e^{-\frac{|\mathbf{k}+\mathbf{k}_0|^2}{2\Delta^2} - i\mathbf{k}\mathbf{a}}, \qquad (1)$$

  consider the time evolution of these states and compute again $\langle\Psi| : \varphi(x)^2 : |\Psi\rangle$. Comment your results.