Quantum Field Theory

Set 6

Exercise 1: P and T for a vector field

Consider a vector field $A^\mu(x)$. In this exercise we will derive its transformation properties under parity and time reversal starting from the Lorentz transformation properties of the field.

Start by considering a field at the origin $x^\mu = 0$. Write the transformation law for this field under a Lorentz transformation and use it to compute the following commutators with the generators of the Lorentz group

$$[J^i, A^\mu(0)] \qquad \text{and} \qquad [K^i, A^\mu(0)]$$

Use this result, together with the transformation properties under P and T for the generators

$$U_P J^i U_P = J^i, \quad U_P K^i U_P = -K^i$$
$$A_T J^i A_T = -J^i, \quad A_T K^i A_T = K^i$$

to prove that

$$U_P A^\mu(0) U_P = \eta_P \mathscr{P}^\mu{}_\nu A^\nu(0)$$
$$A_T A^\mu(0) A_T = -\eta_T \mathscr{P}^\mu{}_\nu A^\nu(0)$$

where

$$\mathscr{P}^\mu{}_\nu = \text{diag}(1, -1, -1, -1).$$

Generalize then the formulas to the case where the field is at some point x^μ by using the transformation properties of the field under translations and

$$U_P P^0 U_P = P^0, \quad U_P P^i U_P = -P^i$$
$$A_T P^0 A_T = P^0, \quad A_T P^i A_T = -P^i$$

where $P^\mu = P^0, P^i$ are the generator of translations.

Exercise 2: Time Reversal of the scalar current

Derive the action of the anti-linear time reversal operator T on the scalar current:

$$A_T^\dagger J^\mu(\vec{x}, t) A_T = \mathscr{P}^\mu{}_\nu J^\nu(\vec{x}, -t)\,, \qquad\qquad J_\mu = i(\phi^\dagger \partial_\mu \phi - \partial_\mu \phi^\dagger\, \phi).$$

Starting from the scalar field transformation property

$$A_T^\dagger \phi(\vec{x}, t) A_T = \eta_T \phi(\vec{x}, -t)$$

Exercise 3: Parity of Dirac Fermions

Recalling the transformation properties of Weyl fermions under parity

$$U^\dagger(P)\, \chi_L(x)\, U(P) = \eta_R \chi_R(x_P),$$
$$U^\dagger(P)\, \chi_R(x)\, U(P) = \eta_L \chi_L(x_P),$$

where $x_P = (t, -\vec{x})$, show that the Dirac action

$$S = \int d^4x \, \left(i\chi_L^\dagger \, \bar{\sigma}^\mu \partial_\mu \, \chi_L + i\chi_R^\dagger \sigma^\mu \partial_\mu \chi_R - m(\chi_R^\dagger \chi_L + \chi_L^\dagger \chi_R) \right)$$

is invariant under parity. Which values of η_R and η_L are allowed? Derive the matrix U_p that describes the transformation properties of a dirac fermion according to the following formula:

$$U^\dagger(P) \, \psi(x) \, U(P) = \eta_P U_P \psi(x_P), \qquad \psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$$

Finally, deduce the transformation properties of all the bilinears of the form $\bar{\psi}\Gamma\psi$, where Γ is an element of the usual basis

$$\Gamma = \{1_4, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \gamma^{\mu\nu}\}.$$