Quantum Field Theory

Set 3

Exercise 1: Physical observables

Consider the Gupta-Bleuler Lagrangian:

$$\mathcal{L}_{GB} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu).$$

- Compute the conserved momentum P_ν through the Noether procedure.
- By working with the algebra of the ladder operators, show that P_ν is a physical observable in the sense that:

  $$[L, P_\nu] \sim \partial_\nu L.$$

  where $L \equiv \partial^\mu A^-_\mu$.

Exercise 2: Propagator of the Gupta-Bleurer Lagrangian

Consider the Gupta-Bleurer Lagrangian with generic coefficient ξ, in presence of an external source J^μ:

$$\mathcal{L}_{GB} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\xi}{2}(\partial_\mu A^\mu)^2 + A^\mu J_\mu$$

- Find the EOM for A^μ, and write it in a following form

  $$\Pi^{\mu\nu} A_\nu = J^\mu$$

  where $\Pi^{\mu\nu}$ is a tensor dependent on ∂_μ and $\eta^{\mu\nu}$
- Invert the EOM:

  $$A^\mu(x) = (\Pi^{-1})^{\mu\nu} J_\nu(x) \tag{1}$$

  where $(\Pi^{-1})^{\mu\nu}$ is called *propagator*.
  (*Hint:* Decompose $\Pi^{\mu\nu}$ into the orthogonal projectors $P_L^{\mu\nu} = \frac{1}{\Box}\partial^\mu\partial^\nu$, $P_T^{\mu\nu} = \eta^{\mu\nu} - \frac{1}{\Box}\partial^\mu\partial^\nu$)
  Is this procedure possible for $\xi = 0$?
- Specialize now to the case $\xi = 1$ and solve for the Green function of the theory $G^{\mu\nu}(x)$:

  $$G^{\mu\nu}(x) = (\Pi^{-1})^\mu{}_\alpha \eta^{\alpha\nu} \delta^4(x) \tag{2}$$

  Use the prescription for going around the poles at $k^0 = \pm|\vec{k}|$ in order to have the *Retarded* Green function
- Use now instead the Feynman prescription, which is obtained by the replacement $k^2 \to k^2 + i\epsilon$ for $\epsilon \to 0^+$. (Do not perform the integral over $\vec{k}$ explicitly).

Optional Exercise: Degrees of freedom of a two-form field

Consider a completely anti-symmetric field $F_{\mu\nu\rho}$.

- Can you determine in which representation of the Lorentz group does this field transform?

The answer to this question is confirmed by the possibility of writing the dual field

$$\tilde{F}_\mu = \frac{1}{3!}\varepsilon_\mu{}^{\nu\rho\sigma}F_{\nu\rho\sigma}. \tag{3}$$

Why? Consider now the particular case where $F_{\mu\nu\rho}$ is written in terms of a 2-index anti-symmetric tensor as

$$F_{\mu\nu\rho}(A) = 3\partial_{[\mu}A_{\nu\rho]}, \tag{4}$$

Note that this is invariant under the gauge transformation

$$A_{\mu\nu} \to A_{\mu\nu} + 2\partial_{[\mu}\omega_{\nu]}. \tag{5}$$

Now, consider the following Lagrangian for this field

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu\rho}(A)F_{\mu\nu\rho}(A). \tag{6}$$

- What are the equations of motion?
- What is the Bianchi identity for the field strength? (Think about Maxwell's electrodynamics).
- How many Gauss laws constraints are there? How many propagating degrees of freedom does the theory describe?

At this point, use equation (3) to re-write the equations of motion and the Bianchi identity in terms of the dual field $\tilde{F}$.

- Realize that the equation of motion allows you to write $\tilde{F}$ in terms of a single scalar field ϕ.
- Rewrite the Bianchi identity and the Lagrangian in terms of ϕ. Can you recognize the result?