Quantum Field Theory

Set 2

Exercise 1 (Optional): The Casimirs of the Poincaré group

- Consider the generators of the Poincaré group, i.e. the translation generators P_μ and the Lorentz generators $J_{\mu\nu}$ (which are antisymmetric in $\mu \leftrightarrow \nu$). Let us write $J^{\mu\nu}$ (in analogy to electromagnetism) as

$$J^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \tag{1}$$

By applying a series of Lorentz transformations to P and J, show that they can be brought to the form

$$J^{\mu\nu} = \begin{pmatrix} 0 & E & 0 & 0 \\ -E & 0 & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & -B & 0 \end{pmatrix} \qquad P^\mu = \begin{pmatrix} P^0 \\ 0 \\ P^2 \\ 0 \end{pmatrix} \tag{2}$$

Hint: Argue using the electromagnetism analogy that one can apply some rotations and boosts in such a way to make $\vec{E} \to (E, 0, 0)$ and $\vec{B} \to (B, 0, 0)$. Argue that after having fixed B and E, there are two boosts left that one can use to set $P^1 \to 0$ and $P^3 \to 0$.

- Since we exhausted all Lorentz transformations to make most components vanish, this constructions shows that there are 4 quantities that are invariant under Lorentz transformations. It is useful to write them in a more covariant manner. To achieve this compute

$$C_1 = J^{\mu\nu} J_{\mu\nu} \quad C_2 = \epsilon_{\mu\nu\rho\sigma} J^{\mu\nu} J^{\rho\sigma} \quad C_3 = P^\mu P_\mu \quad C_4 = W^\mu W_\mu \tag{3}$$

using (2). Conclude that C_1, C_2, C_3 and C_4 contain as much information as E, B, P^0, P^2 but have the advantage of being manifestly Lorentz invariant.

- Finally, show that C_1 and C_2 do not commute with translations, while C_3 and C_4 do. Thus, C_3 and C_4 are the two Casimirs of the Poincaré group.

Exercise 2: Maxwell's equations and transverse components

- Show the the quantities $P_\perp^{ij} = \left(\delta^{ij} - \frac{\partial^i \partial^j}{\nabla^2}\right)$, $P_L^{ij} = \frac{\partial^i \partial^j}{\nabla^2}$ are orthogonal projectors on the space of three-dimensional vector fields:

$$(P_L)^{ij}(P_\perp)^{jk} = 0\,, \qquad (P_L)^{ij}(P_L)^{jk} = (P_L)^{ik}\,, \qquad (P_\perp)^{ij}(P_\perp)^{jk} = (P_\perp)^{ik}\,, \qquad P_L^{ij} + P_\perp^{ij} = \delta^{ij}$$

- Decompose the electric and magnetic field into longitudinal and transverse parts, $\vec{E} = \vec{E}_L + \vec{E}_\perp$, $\vec{B} = \vec{B}_L + \vec{B}_\perp$, where:

$$E_\perp^i = P_\perp^{ij} E^j\,, \qquad E_L^i \equiv P_L^{ij} E^j$$

Using the Bianchi identity for the field strength, namely $\epsilon_{\mu\nu\rho\sigma}\partial^\mu F^{\rho\sigma} = 0$, show that the number of degrees of freedom encoded in $F_{\mu\nu}$ is **3**. In particular, prove that $\vec{B}_L = 0$, while $\vec{B}_\perp$ can be expressed in terms of $\vec{E}_\perp$ only. Finally, consider Maxwell's inhomogeneous equations to show that $\vec{E}_L$ is completely fixed by the charge density. Thus, the *number of dynamical degrees of freedom* is only **2**.

- Now solve the Maxwell equation for A^0 and substitute the solution into the remaining non trivial equations. Show that the result is a wave equation for the transverse components of the gauge potential and that the longitudinal component decouples completely.

Exercise 3: Quantization in Coulomb gauge

Consider the Lagrangian of a massless vector field interacting with a matter source J^μ:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu \ ,$$

where $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$. In this exercise we show how to quantize the massless vector field $A_\mu = (A_0, -\vec{A})$ in the Coulomb gauge

$$\vec{\nabla} \cdot \vec{A} = 0 \tag{4}$$

- Having in mind that the gauge condition (4) is not Lorentz invariant, show that the Lagrangian can be rewritten explicitly in terms of $A_\mu = (A_0, -\vec{A})$ as:

$$\mathcal{L} = \frac{1}{2}\left(\dot{\vec{A}} + \vec{\nabla}A_0\right)^2 - \frac{1}{2}\left(\vec{\nabla}\wedge\vec{A}\right)^2 - J^0 A_0 + \vec{J}\cdot\vec{A}$$

- Show that, thanks to the gauge invariance of the theory ($A_\mu(x) \longrightarrow A_\mu(x) - \partial_\mu\Lambda(x)$), we can always impose the constraint $\vec{\nabla}\cdot\vec{A} = 0$ (i.e. we can always find a Λ such that $\vec{A}$ satisfies that constraint).
- Compute the conjugate momenta $\Pi_i = \frac{\partial\mathcal{L}}{\partial\dot{A}^i}$ and $\Pi^0 = \frac{\partial\mathcal{L}}{\partial\dot{A}_0}$. What about Π^0? Is it consistent with the usual canonical commutation relations $[A^\mu(\vec{x},t), \Pi_\nu(\vec{y},t)] = i\delta^\mu_\nu\delta^3(\vec{x}-\vec{y})$?
- Show that from the EOM of A_0 in Coulomb Gauge can be solved to give:

$$A_0 = \int d^3y \frac{J^0(\vec{y},t)}{4\pi|\vec{x}-\vec{y}|}$$

  Hint: recall that the Green function of the Laplacian is given by:

$$\vec{\nabla}^{-2}(\vec{x}-\vec{y}) = -\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{\vec{k}^2} = \frac{-1}{4\pi|\vec{x}-\vec{y}|}$$

- The remaining canonical variable are $(A^i, \Pi_i) \equiv (\vec{A}, \vec{\Pi})$. Show that as a consequence of the EOM for A_0 the cononical momenta must obey the relation:

$$-\partial^i\Pi_i = \vec{\nabla}\cdot\vec{\Pi} = J^0.$$

  Show then that the modified commutation relation

$$[A(\vec{x},t)^i, \Pi(\vec{y},t)_j] = i\delta^i_j\delta^3(\vec{x}-\vec{y}) - i\partial_i^{(x)}\partial_j^{(y)}\frac{1}{4\pi|\vec{x}-\vec{y}|} = i\delta^i_j\delta^3(\vec{x}-\vec{y}) - i\frac{\partial_i^{(x)}\partial_j^{(x)}}{\vec{\nabla}_x^2}\delta^3(\vec{x}-\vec{y})$$

  is consistent with the gauge choice $\vec{\nabla}\cdot\vec{A} = 0$ and with the constraint $\vec{\nabla}\cdot\vec{\Pi} = J^0$.
  Hint: notice that $[\vec{A}, J^0] = 0$, since J^μ depends only on the *matter* degrees of freedom.

- Compute the Hamiltonian $\mathcal{H} = \dot{A}^i\Pi_i - \mathcal{L}$ using the expression for $\vec{\Pi}$

$$\vec{\Pi} = \vec{\nabla}A^0 + \dot{\vec{A}}.$$

- Define the solenoidal part of $\vec{\Pi}$ as

$$\vec{\Pi}_\perp = \vec{\Pi} - \vec{\nabla}A_0.$$

  Show that this satisfies the constraint $\vec{\nabla}\cdot\vec{\Pi}_\perp = 0$. Show that the Hamiltonian can be written in term of this as:

$$\mathcal{H} = \frac{1}{2}\vec{\Pi}_\perp^2(\vec{x},t) + \frac{1}{2}\left(\vec{\nabla}\wedge\vec{A}(\vec{x},t)\right)^2 + \frac{1}{2}\int d^3y\frac{J^0(\vec{x},t)J^0(\vec{y},t)}{4\pi|\vec{x}-\vec{y}|} - \vec{J}(\vec{x},t)\cdot\vec{A}(\vec{x},t)$$

Exercise 4: Energy momentum tensor

Consider the following Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\lambda}{2}(\partial_\rho A^\rho)^2.$$

- Find the energy momentum tensor using the standard procedure and show explicitly that it is conserved by imposing the equations of motion.
- Discuss the gauge transformation properties of $T^\mu_{\ \nu}$ for $\lambda = 0$. What about the associated charges?
- Improve the energy momentum tensor you found for $\lambda = 0$ by adding a term $F^{\mu\rho}\partial_\rho A_\nu$. Show that the new tensor is still conserved and gives rise to the same charges as before, but is now also symmetric, traceless and gauge invariant.
- What happens if $\lambda \neq 0$? Do currents and charges depend on λ?