Quantum Field Theory II

Set 1

Exercise 1: The Pauli–Lubanski (pseudo)vector

The Poincare group has two casimirs, P^2 and W^2, where P^μ and W^μ are respectively the momentum and the Pauli-Lubanski vector

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma \,.$$

- Show that this definition is equivalent to

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu J_{\rho\sigma} \,.$$

- Calculate and express the result of the following commutators in terms of the Pauli-Lubanski vector
  1. $P_\mu W^\mu$,
  2. $[P^\mu, W^\nu]$,
  3. $[J^{\mu\nu}, W^\rho]$ (Hint: Use the identity $\eta^{\mu\nu}\epsilon^{\rho\alpha\beta\sigma} = \eta^{\mu\rho}\epsilon^{\nu\alpha\beta\sigma} + \eta^{\mu\alpha}\epsilon^{\rho\nu\beta\sigma} + \eta^{\mu\beta}\epsilon^{\rho\alpha\nu\sigma} + \eta^{\mu\sigma}\epsilon^{\rho\alpha\beta\nu}$),
  4. $[W^\mu, W^\nu]$.
- Show that W^2 is a Casimir of the Poincare group.

Exercise 2: The spin of Dirac Fermion states

In this exercise you will show that Dirac Fermion states have spin 1/2.

Consider the state of a single Dirac fermion

$$|\Psi\rangle = \int d^3x f_\alpha(x) \psi^\dagger_\alpha(x) |0\rangle \,. \tag{1}$$

Here ψ is the Dirac field and f is a generic wave packet. Using the mode expansion of the field

$$\psi_\alpha(x) = \sum_r \int d\Omega_p \, u(p,r) e^{-ipx} a_r(p) + v(p,r) e^{ipx} b^\dagger_r(p), \tag{2}$$

rewrite the state as

$$|\Psi\rangle = \sum_r \int d\Omega_p F(p,r) |p,r\rangle \,, \tag{3}$$

where

$$F(p,r) = u^\dagger_\alpha(p,r) \tilde{f}_\alpha(p), \tag{4}$$

and

$$\tilde{f}_\alpha(p) = \int d^3x e^{ipx} f_\alpha(x). \tag{5}$$

Consider the angular momentum operator you computed in Exercise 14.2

$$J^i = \int d^3x \, \psi^\dagger_\alpha(t,\vec{x}) \Big([\vec{x} \wedge (-i\vec{\nabla})] \delta_{\alpha\beta} + (\Sigma)_{\alpha\beta}/2 \Big) \psi_\beta(t,\vec{x}). \tag{6}$$

Looking at this operator, one might be tempted to identify the first term as the orbital angular momentum and the second one as the spin. However, this is not true and the action of the two operators on the state $|\Psi\rangle$ will mix. Show that

$$\vec{J}\,|\Psi\rangle = \sum_{r,r'} \int d\Omega_p \Big((-i\vec{p}\wedge\vec{\partial}_p)\delta_{rr'} + (\vec{\sigma})_{rr'}/2 \Big) F(p,r')\,|p,r\rangle\,. \tag{7}$$

Only at this point you can identify the first term as the representation of the orbital angular momentum and the second one as the spin acting on the wave function $F(p,r)$. Choosing a state with zero orbital angular momentum, realize that Dirac fermion states have spin 1/2.

Hint: You may find convenient to first compute the commutator of the angular momentum operator $\vec{J}$ with the field $\psi^\dagger$

$$[\vec{J},\psi^\dagger_\alpha(x)] = \psi^\dagger_\beta(t,\vec{x})\Big(\vec{x}\wedge(i\overleftarrow{\nabla}) + \vec{\Sigma}/2\Big)_{\beta\alpha}. \tag{8}$$

Then, you should prove

$$u^\dagger(p,r)\vec{\Sigma}/2 = (-i\vec{p}\wedge\vec{\partial}_p)u^\dagger(p,r) + \sum_{r'}(\vec{\sigma})_{rr'}/2\,u^\dagger(p,r'). \tag{9}$$

Exercise 3: Transformation of the boost generators under translations

Starting from the group composition rule:

$$g(\Lambda_1,a_1)g(\Lambda_2,a_2) = g(\Lambda_1\Lambda_2,\Lambda_1 a_2 + a_1), \tag{10}$$

with $g(\Lambda,a)\in ISO(3,1)$, prove the following transformation of $J^{\mu\nu}$:

$$e^{-iaP}J^{\mu\nu}e^{iaP} = J^{\mu\nu} + P^\nu a^\mu - P^\mu a^\nu. \tag{11}$$

where a^μ is a translation vector.