Quantum Field Theory

Set 11

Exercise 1: Scattering $2 \to 2$ in $\lambda\phi^3$

Starting from the matrix element

$$|\mathcal{M}(p_a, p_b \to p_c, p_d)|^2 \equiv |\mathcal{M}|^2 = \lambda^4 \left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right]^2,$$

where m is the mass of the scalar particle in the $\lambda\phi^3$ theory, derive the expression for the differential cross section $\frac{d\sigma}{d\cos\theta}$. Here we call θ the angle between $\vec{p}_a$ and $\vec{p}_c$.

By dimensional analysis (i.e. considering all dimensionless quantities $\simeq O(1)$, and $t \simeq u \simeq s$), guess the behavior of the total cross section $\sigma(s)$ in the ultrarelativistic limit $s \gg 4m^2$. Then, in the same limit, compute the dominant term of the cross section by integrating the expression found before. Compare with your guess and explain the differences.

Show that the dominant contribution to the integral $\sigma = \int \frac{d\sigma}{d\cos\theta}\, d\cos\theta$ is given by the region where $\theta \simeq 0$ (or $\theta \simeq \pi$) and give an estimate of $\theta_{\max}$ representing the maximal angular distance from $\theta = 0$ (or $\theta = \pi$) which gives a relevant contribution to the integral. Compute the transverse momentum $(p_\perp)_{\max}$ associated to a particle scattered at angle $\theta_{\max}$. Find the impact parameter $b_{\min}$ associated to $(p_\perp)_{\max}$ and conclude that the dominant contribution to the cross section is given by particles interacting at large distances.

Consider now the opposite case of hard scattering, where the transverse momentum $p_\perp$ is large and the angle θ is close to the perpendicular. Estimate the contribution to the cross section from big angles (for example $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$). Comment the result.

Finally consider the opposite limit, $s - 4m^2 \ll m^2$, and compute the total cross section in this nonrelativistic approximation.

Application: consider a ϕ particle with energy $E \gg m^2$ crossing a box of length ℓ full of target particles with density ρ. For simplicity, suppose also that the incoming and target particles are distinguishable. Compute the mean energy loss $\langle \Delta E \rangle$ per crossed length dx as a function of the energy of the incoming particle.
Hint: suppose that $\langle \Delta E \rangle$ is mainly determined by scattering at small angles θ. Justify this a posteriori.

Exercise 2: Traces of Dirac matrices

Making use of the anticommutation properties of the Dirac matrices,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu},$$

show that:

$$\begin{aligned} &\mathrm{Tr}[\gamma^\mu \gamma^\nu] = 4\eta^{\mu\nu}\,, \\ &\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\rho\nu}). \end{aligned}$$

Recalling the definition of $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ and the anticommutation property $\{\gamma^5, \gamma^\mu\} = 0$, show that

$$\begin{aligned} &\mathrm{Tr}[\text{odd number of } \gamma\text{'s}] = 0, \\ &\mathrm{Tr}[\gamma^5\gamma^\mu \gamma^\nu] = 0, \\ &\mathrm{Tr}[\gamma^5 \cdot (\text{odd number of } \gamma\text{'s})] = 0, \\ &\mathrm{Tr}[\gamma^5\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = -4i\epsilon^{\mu\nu\rho\sigma}. \end{aligned}$$

(Hint: for the first of the above identities insert $\gamma^5\gamma^5 = 1$ in the trace).