Quantum Field Theory

Set 10

Exercise 1: Decay of a massive scalar particle

Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}M^2\Phi^2 - \frac{\lambda}{2}\phi^2\Phi,$$

with $M > 2m$ and λ a real, positive parameter. Using the Lagrangian above, compute the lifetime τ of the particle of mass M.

To perform the computation, use the relation between $\mathcal{M}_{\Phi\to 2\phi}$, entering the definition of the decay width, and the S-matrix element, and also the explicit expression of the latter in terms of the matrix element of the interaction Hamiltonian between initial and final states. In order to evaluate this, expand the scalar fields in terms of creation and annihilation operators.

Exercise 2: Scattering in $\lambda\phi^4$

Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4!}\phi^4.$$

Using the above Lagrangian and Wick's theorem, compute the matrix element $\mathcal{M}_{2\phi\to 2\phi}$ and the total cross section for the scattering process $\phi\phi \to \phi\phi$.

Exercise 3 (Optional): Lippman Schwinger equation and evolution in the interaction picture

Consider the evolution operator in interaction picture:

$$U(t,t') = e^{iH_0 t}e^{-iH(t-t')}e^{-iH_0 t'}.$$

It obeys the equation

$$i\frac{\partial U(t,t')}{\partial t} = H_I(t)U(t,t'), \tag{1}$$

where $H_I(t) = e^{iH_0 t}(H - H_0)e^{-iH_0 t}$.

- Show that the formal solution of (1) is

$$U(t,t') = \mathbb{1} - i\int_{t'}^{t} d\tau H_I(\tau)U(\tau,t')d\tau. \tag{2}$$

- Show that:

$$U(0,-\infty) = \Omega_+ \qquad \textbf{and} \qquad U(t,-\infty) = e^{iH_0 t}\Omega_+ e^{-iH_0 t},$$

  where Ω_+ is the incoming Moeller operator.

Consider now an *in* state

$$|\psi_\alpha^+\rangle = \Omega_+ |\phi_\alpha\rangle .$$

Show that eq. (2), when applied on an free state, is equivalent to Lippman-Schwinger equation:

$$|\psi_\alpha^+\rangle = |\phi_\alpha\rangle + \frac{1}{E_\alpha - H_0 + i\epsilon}(H - H_0)|\psi_\alpha\rangle .$$