S-matrix in perturbation theory

△ Simplest examples of interacting theories

· $\phi^4 \Rightarrow$

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2}_{\mathcal{L}_0} \underbrace{- \frac{\lambda}{4!}\phi^4}_{\mathcal{L}_I}$$

$$H \equiv \int \mathcal{H} d^3x \qquad \mathcal{H} = \underbrace{\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2}_{H_0} \underbrace{+ \frac{\lambda}{4!}\phi^4}_{H_I}$$

- The identification of the split between H_0 and H_I is however a delicate matter

- LS procedure only well defined with proper split
- The issue arises beyond leading order in perturbation theory ... and thus beyond this course

- Basic physics: H_I generically "renormalizes" properties of single particle states in H_0 (ex their mass)

- Proper split

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I = (\mathcal{L}_0 + \Delta\mathcal{L}) + (\mathcal{L}_I - \Delta\mathcal{L})$$

naive split

$$\equiv \mathcal{L}_0' + \mathcal{L}_I'$$

$$\Delta\mathcal{L} = \frac{\delta Z}{2}(\partial_\mu\phi)^2 + \frac{\delta m^2}{2}\phi^2$$

wave function renormalization

mass renormalization

Ex QED

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi - m\bar{\psi}\psi$$

$$= \underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi}_{\mathcal{L}_0} \underbrace{- e\bar{\psi}\gamma^\mu\psi A_\mu}_{\mathcal{L}_I}$$

- $\mathcal{H}_I = -\mathcal{L}_I = e\bar{\psi}\gamma^\mu\psi A_\mu$

- In general $\mathcal{H}_I \neq -\mathcal{L}_I$

$\mathcal{H}_I \neq$ Lorentz scalar

- However it <u>magically</u> happens that the formulae we shall derive apply also to these cases (with some clever adjustments)

Time ordered product

<u>def</u> $$T\left(O_1(x_1)\cdots O_n(x_n)\right) \equiv$$

$$\equiv (\nu)^{n_\pi} O_{\pi(1)}\left(x_{\pi(1)}\right)\cdots\cdots O_{\pi(n)}\left(x_{\pi(n)}\right)$$

- $\pi \equiv$ permutation such that $t_{\pi(1)} > t_{\pi(2)} \cdots > t_{\pi(n)}$
- n_π order of π
- $\nu = 1$ for bosons
- $\nu = -1$ for fermions

- bosonic operators commute within T
- fermionic " anticommute "

Dyson Series

$$S = \Omega_-^\dagger \Omega_+ = \lim_{\substack{T \to \infty \\ T_0 \to -\infty}} \Omega^\dagger(T)\,\Omega(T_0) \equiv \lim_{\substack{T \to \infty \\ T_0 \to \infty}} U(T, T_0)$$

$$U(T, T_0) = e^{iH_0 T}\, e^{-iH(T - T_0)}\, e^{-iH_0 T_0}$$

$$\frac{dU}{dT} = i\, e^{iH_0 T} (H_0 - H)\, e^{-iH(T-T_0)}\, e^{-iH_0 T_0}$$

interaction picture

$$= -i\, e^{iH_0 T} H_I\, e^{-iH_0 T}\, U(T, T_0) \equiv -i\, H_I(T)\, U(T, T_0)$$

$$\begin{cases} \dfrac{d}{dT} U(T, T_0) = -i\, H_I(T)\, U(T, T_0) \\ \\ U(T_0, T_0) = \mathbb{1} \end{cases}$$

$$\int_{T_0}^{T} \frac{d}{dT} U(T, T_0) = U(T, T_0) - U(T_0, T_0)$$
$$= U(T, T_0) - \mathbb{1}$$

integrating

$$\boxed{U(T, T_0) = \mathbb{1} - i \int_{T_0}^{T} H_I(t)\, U(t, T_0)\, dt}$$

- this can be solved by iteration treating H_I as a perturbation

$$\Downarrow$$

Notice

Aside !!

$$\Omega_+ = U(0,-\infty) = \mathbb{1} - i\int_{-\infty}^{0} H_I(t)\, U(t,-\infty)\, dt$$

is time representation of Lippmann-Schwinger

. Can see that by using $U(t,-\infty) = e^{iH_0 t}\,\Omega_+\, e^{-iH_0 t}$

. acting on $|\phi_\alpha\rangle$

- Left as exercise

$$U(T,T_0) = \mathbb{1} - i\int_{T_0}^{T} H_I(t)\,dt + (-i)^2\int_{T_0}^{T} H_I(t_1)\int_{T_0}^{t_1} H_I(t_2)\,dt_1\,dt_2 +$$

$$+ (-i)^3\int_{T_0}^{T} H_I(t_1)\int_{T_0}^{t_1} H_I(t_2)\int_{T_0}^{t_2} H_I(t_3)\,dt_1\,dt_2\,dt_3 + \dots$$

$$= \mathbb{1} - i\int_{T_0}^{T} H_I(t)\,dt + \frac{(-i)^2}{2}\int_{T_0}^{T} T\big(H_I(t_1)\,H_I(t_2)\big)\,dt_1\,dt_2 +$$

$$+ \frac{(-i)^3}{3!}\int_{T_0}^{T} T\big(H_I(t_1)\,H_I(t_2)\,H_I(t_3)\big)\,dt_1\,dt_2\,dt_3 + \dots$$

$$= T\exp\int_{T_0}^{T} -i\,H_I(t)\,dt$$

$$S = \lim_{\substack{T\to\infty \\ T_0\to-\infty}} U(T,T_0) = T\exp\int_{-\infty}^{\infty} -i H_I(t)\,dt$$

$$= T\exp\int i\mathcal{L}(x,t)\,d^3x\,dt$$

- This Dyson's series

- In the simplest cases $H_I = -\int \mathcal{L}_I\, d^3x \equiv \int \mathcal{H}_I\, d^3x$

$$\Rightarrow \quad S = T\exp\int i\mathcal{L}_I(x)\,d^4x$$

a manifestly Lorentz invariant

- **Ex** ϕ^4: $\quad H_I(t) = e^{iH_0 t} \int -\frac{\lambda}{4!}\phi^4(0,x)\, d^3x \; e^{-iH_0 t}$

$$\boxed{S = U^{-1}(\Lambda)\, S\, U(\Lambda)}$$

$$e^{iH_0 t}\phi^4(0,x)e^{-iH_0 t} = \left[e^{iH_0 t}\phi(0,x)e^{-iH_0 t}\right]^4 = \left[\phi(t,x)\right]^4$$

$$= -\frac{\lambda}{4!}\int \phi^4(t,x)\, d^3x$$

$$\Rightarrow \quad -i\int H_I(t)\,dt = i\frac{\lambda}{4!}\int \phi^4(t,x)\underbrace{d^3x\,dt}_{d^4x}$$

$$x' \equiv \Lambda^{-1}x$$

$$U(\Lambda)^{+}\int \phi^4(x)\, d^4x \; U(\Lambda) = \int \phi^4(\Lambda^{-1}x)\, d^4x$$

$$\Lambda^{-1}x \equiv x' \qquad \equiv \int \phi^4(x')\, d^4x'$$

- Lorentz invariance with more care

assume $S = \exp i \int \mathcal{L}\, d^4x$

$U_\Lambda^+ S U_\Lambda = \exp i \int U_\Lambda^+ \mathcal{L}_I(x) U_\Lambda\, d^4x$

$= \exp i \int \mathcal{L}_I(\Lambda^{-1}x)\, d^4x$

$= S$

$$S = T \exp i \int \mathcal{L}_I(x)\, d^4x$$

- if it wasn't for T-ordering, Lorentz would be obvious: $\mathcal{L}_I \equiv$ Lorentz scalar built out of Heisenberg picture fields of free theory

$$U_\Lambda^+ \mathcal{L}_I(x) U_\Lambda = \mathcal{L}_I(\Lambda^{-1}x)$$

$x' = \Lambda^{-1}x$

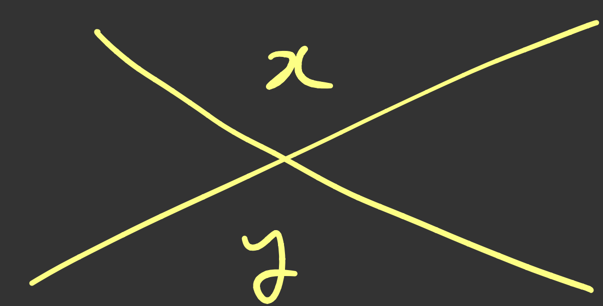
$$U_\Lambda^+ \left(\int \mathcal{L}_I(x)\, d^4x \right) U_\Lambda = \int \mathcal{L}_I(\Lambda^{-1}x)\, d^4x = \int \mathcal{L}_I(x')\, d^4x'$$

- T-ordering makes things more complicated

- Consider $T\left(\int \mathcal{L}_I(x)\mathcal{L}_I(y)\, d^4x\, d^4y\right)$

- Two options for (x, y)

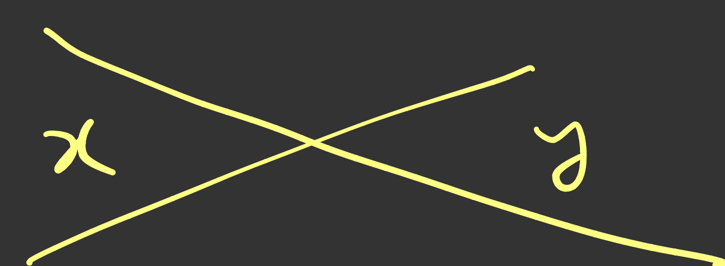
  - time-like



$(x-y)^2 > 0$

$\Longrightarrow$ T-ordering is invariant

  - space-like



$(x-y)^2 < 0$

$\Longrightarrow$ T-ordering non-invariant

- ordering of space-like events is problematic

... unless $[\mathcal{L}(x), \mathcal{L}(y)] = 0$ for $x \times y$

- This condition, that physical operators commute outside light-cone, corresponds to the basic notion of <u>causality</u>

- it is crucial to ensure relativistic invariance

$$\text{relativity} \Longleftrightarrow \text{causality}$$

$$S = \mathbb{1} + S^{(1)} + S^{(2)} + S^{(3)} + \dots$$

$$S^{(2)} = \frac{(i)^2}{2} \underbrace{T \int \mathcal{L}_I(x) \mathcal{L}_I(y) \, d^4x \, d^4y}_{\int d^4x \, d^4y \left(\theta(t_x - t_y) \mathcal{L}(x) \mathcal{L}(y) + \theta(t_y - t_x) \mathcal{L}(y) \mathcal{L}(x) \right)}$$

want to check $U_\Lambda^{+} S^{(2)} U_\Lambda = S^{(2)}$

$$U_\Lambda^{+} S^{(2)} U_\Lambda = \frac{(i)^2}{2} \int d^4x \, d^4y \left(\theta(t_x - t_y) U_\Lambda^{+} \mathcal{L}(x) \mathcal{L}(y) U_\Lambda + \theta(t_y - t_x) U_\Lambda^{+} \mathcal{L}(y) \mathcal{L}(x) U_\Lambda \right)$$

$$= \frac{(i)^2}{2} \int d^4x\, d^4y \left[\theta(t_x - t_y)\, \mathcal{L}(\Lambda^{-1}x)\, \mathcal{L}(\Lambda^{-1}y) + \theta(t_y - t_x)\, \mathcal{L}(\Lambda^{-1}x)\, \mathcal{L}(\Lambda^{-1}y) \right]$$

$$\int d^4x\, d^4y = \int d^4x\, d^4y + \int d^4x\, d^4y$$

$$x' = \Lambda x$$
$$y' = \Lambda y$$

$$= \int d^4x'\, d^4y' + \int d^4x'\, d^4y'$$

$$\int_{\underset{\cdot}{\dot{\times}}} d^4x\, d^4y \;\; \theta(t_x - t_y)\, \mathcal{L}(\Lambda^{-1}x)\, \mathcal{L}(\Lambda^{-1}y) + \theta(t_y - t_x)\, \mathcal{L}(\Lambda^{-1}y)\, \mathcal{L}(\Lambda^{-1}x)$$

$$t_x - t_y > 0 \Leftrightarrow t_{\Lambda^{-1}x} - t_{\Lambda^{-1}y} > 0$$
$$< 0 \Leftrightarrow \qquad < 0$$

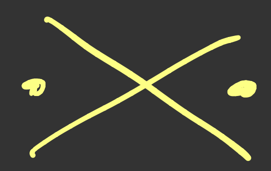
$$\theta(t_{\Lambda^{-1}x} - t_{\Lambda^{-1}y}) \qquad\qquad \theta(t_{\Lambda^{-1}y} - t_{\Lambda^{-1}x})$$

$$= \int_{\underset{\cdot}{\dot{\times}}} d^4x\, d^4y \;\; \theta(t_{\Lambda^{-1}x} - t_{\Lambda^{-1}y})\, \mathcal{L}(\Lambda^{-1}x)\, \mathcal{L}(\Lambda^{-1}y) \; \cdots$$

$$\Lambda^{-1}x \equiv x' \qquad\qquad \Lambda^{-1}y = y'$$

$$= \int_{\underset{\cdot}{\dot{\times}}} d^4x'\, d^4y' \;\; \theta(t_{x'} - t_y)\, \mathcal{L}(x')\, \mathcal{L}(y') + \cdots$$

$$\int d^4x\, d^4y \quad \theta(t_x - t_y)\mathcal{L}(x)\mathcal{L}(y) + \theta(t_y - t_x)\mathcal{L}(y)\mathcal{L}(x)$$



$$= \int d^4x\, d^4y \quad \mathcal{L}(x)\mathcal{L}(y)$$

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