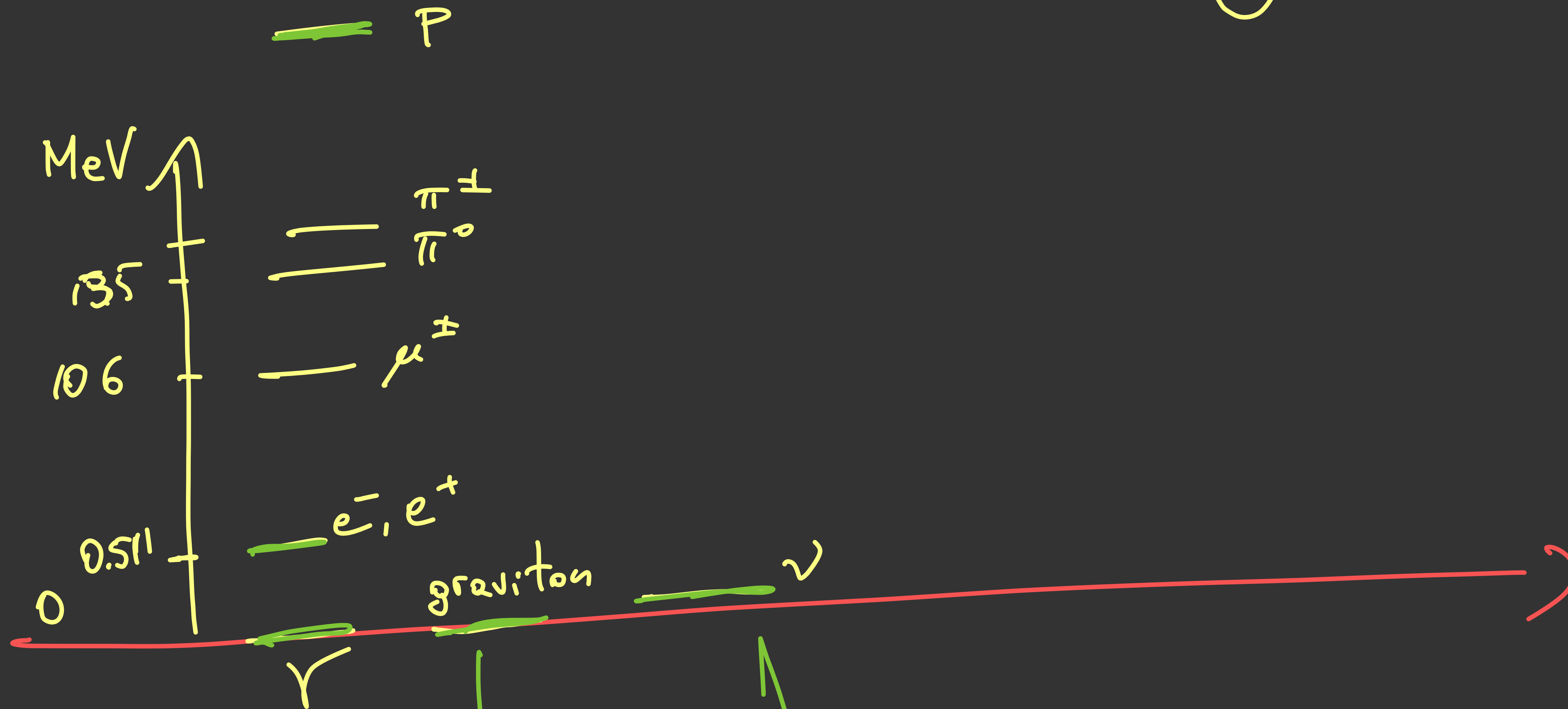
Quantum ElectroDynamics (QED)



incredibly weakly coupled

very weakly coupled

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}\left(i\not{D} - m\right)\psi$$

$$\not{D} = \gamma^{\mu}\left(\partial_{\mu} + ieA_{\mu}\right)$$

$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}\left(i\not{\partial} - m\right)\psi$$

$$\mathcal{L}_{int} = -e A^{\mu} \bar{\psi} \gamma_{\mu} \psi \qquad \Bigg| \; \frac{e^2}{4\pi} = \alpha \sim \frac{1}{137}$$

Propagators

- photon

$$\overbracket{A_\mu(x)\, A_\nu(y)} = \langle 0| T(A_\mu(x)\, A_\nu(y))|0\rangle \cdot \mathbb{1}$$

- $$A_\mu = \int d\Omega_k \left(e^{-ikx}\, a_\mu(k) + e^{ikx}\, a_\mu^+(k) \right)$$

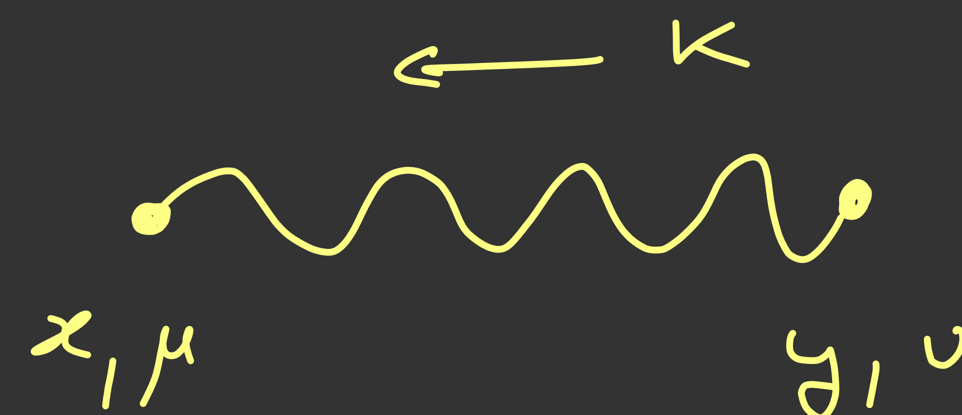
- $$\langle 0|\, a_\mu(k)\, a_\nu^+(k')\,|0\rangle = -\eta_{\mu\nu}\,(2\pi)^3\, 2k^0\, \delta^3(\vec{k}-\vec{k}')$$

$$\Rightarrow \quad \overbracket{A_\mu\, A_\nu} = -\eta_{\mu\nu}\, G_F(x-y,\, m=0)$$

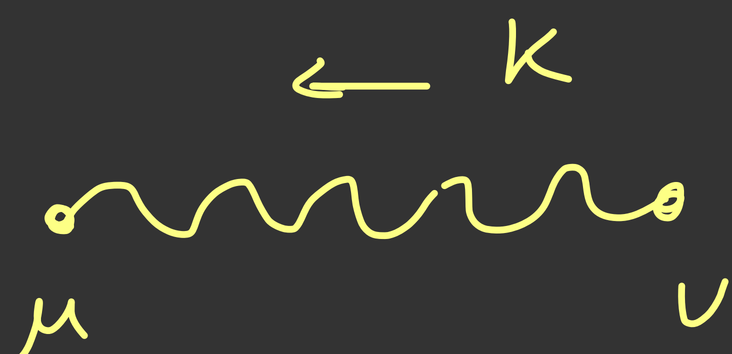
- $\langle 0 | T(A_\mu(x) A_\nu(y)) | 0 \rangle \equiv G_{\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \frac{-i\eta_{\mu\nu}\, e^{-ik(x-y)}}{k^2 + i\varepsilon}$

synthetically

$\langle x, \mu | y, \nu \rangle \Longrightarrow$



- in momentum space



$= \frac{-i\eta_{\mu\nu}}{k^2 + i\varepsilon}$

- Fermion (electron & positron) $\psi_\alpha(x)\bar\psi_\beta(y)$

$$S_{\alpha\beta}(x-y) = \langle 0| T(\psi_\alpha(x)\bar\psi_\beta(y))|0\rangle =$$

$$= \theta(x^0-y^0)\langle 0|\psi_\alpha(x)\bar\psi_\beta(y)|0\rangle - \theta(y^0-x^0)\langle 0|\bar\psi_\beta(y)\psi_\alpha(x)|0\rangle$$

- in exercise set 23 we proved

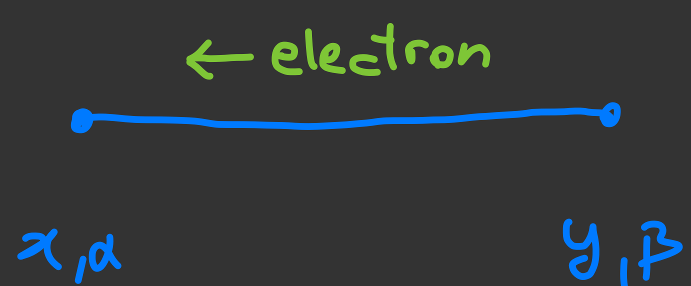
$$(i\not\partial_x - m)_{\alpha\gamma}\, S_{\gamma\beta}(x-y) = i\,\delta^4(x-y)\,\delta_{\alpha\beta}$$

$$(i\not\partial - m)\cdot S = i\,\mathbb{1} \Longrightarrow S = i\cdot(i\not\partial - m)^{-1}$$

$$\Rightarrow S_{\alpha\beta}(x-y) = \int \frac{d^4p}{(2\pi)^4}\left(\frac{i}{\not{p} - m + i\varepsilon}\right)_{\alpha\beta} \cdot e^{-ip(x-y)}$$

$$= \int \frac{d^4p}{(2\pi)^4}\,\frac{i(\not{p} + m)_{\alpha\beta}}{p^2 - m^2 + i\varepsilon}\, e^{-ip(x-y)}$$

$$\theta(x^0-y^0)\langle \psi_\alpha(x)\bar{\psi}_\beta(y)\rangle - \theta(y^0-x^0)\langle \bar{\psi}_\beta(y)\psi_\alpha(x)\rangle$$



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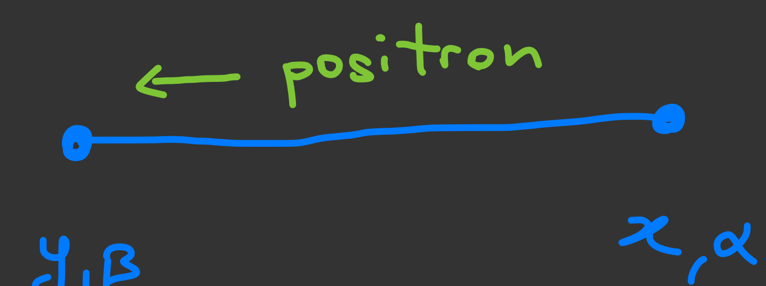
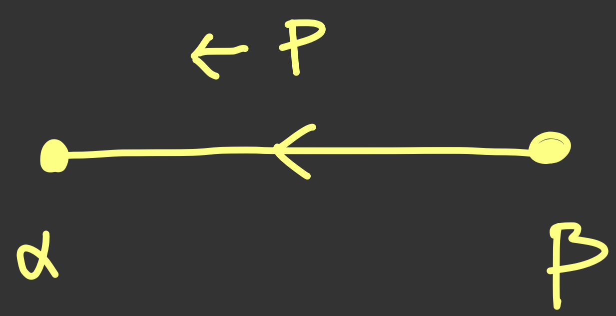


$\equiv$

- in momentum space



$$= \frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\varepsilon}$$

External lines

- **photon**: state is characterized by momentum and spin

$$|k,\varepsilon\rangle \equiv \varepsilon^\mu a_\mu^\dagger(k)|0\rangle \qquad \begin{cases} \varepsilon_\mu^* \varepsilon^\mu = -1 \\ k^\mu \varepsilon_\mu = 0 \end{cases}$$

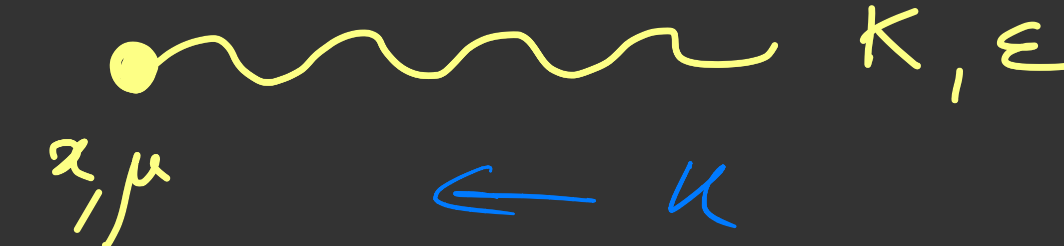
- n-photons

$$|(k_1,\varepsilon_1),\ldots,(k_n,\varepsilon_n)|0\rangle$$

- incoming

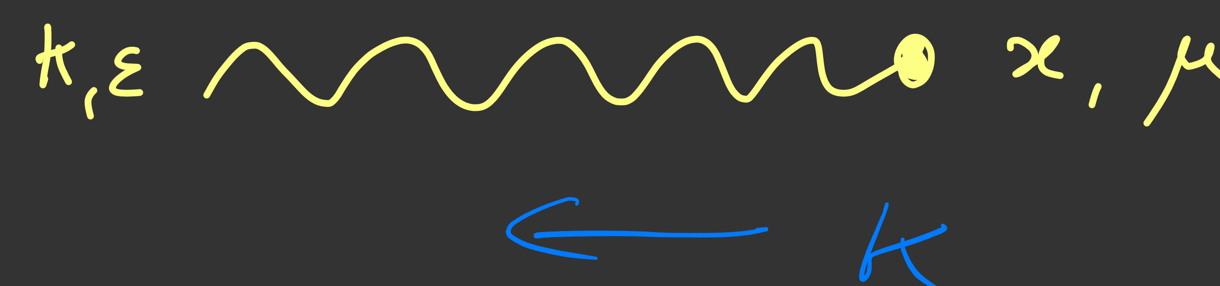
$$A_\mu(x)|k,\varepsilon\rangle \equiv \langle 0|A_\mu(x)|k,\varepsilon\rangle = e^{-ikx}\,\varepsilon_\mu(k)$$

$$\langle x,\mu|k,\varepsilon\rangle$$

$\equiv$ 

- outgoing

$$\langle k,\varepsilon|A_\mu(x)|0\rangle \equiv \langle k,\varepsilon|A_\mu(x)|0\rangle = e^{ikx}\,\varepsilon^*_\mu(k)$$

$\equiv$ 

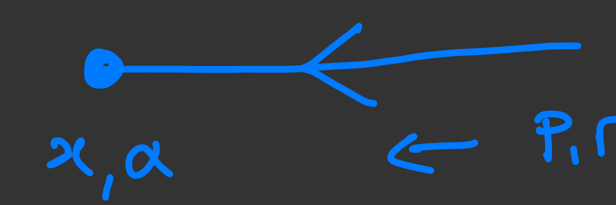
- **fermion** : momentum, spin ($Q = \pm 1$, particle and anti particle)

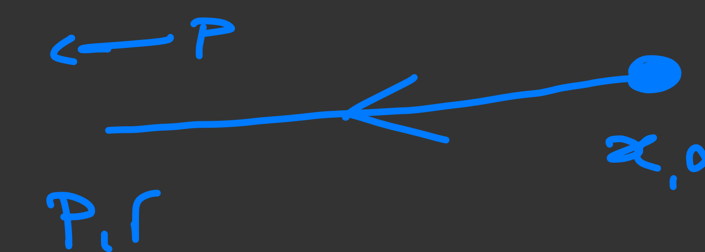
$$\psi_\alpha(x) = \int d\Omega_p \sum_{r=1}^{2} \left(e^{-ipx} u_\alpha(r,p)\, a(r,p) + e^{ipx} v_\alpha(r,p)\, b^+(r,p) \right)$$

- particle state (electron) : $|p, r\rangle \equiv a^+(r,p)|0\rangle$

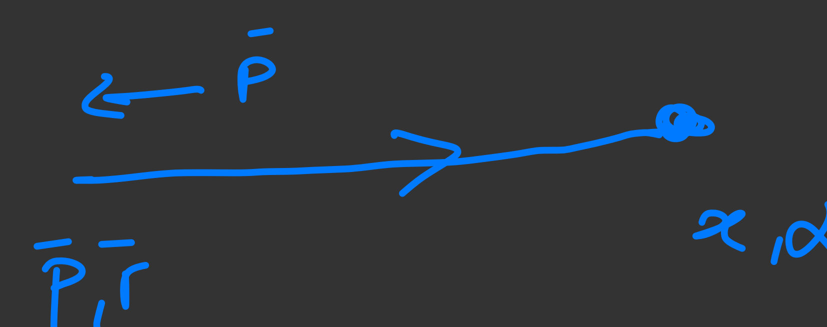
- anti-particle state (positron) : $|\bar{p}, \bar{r}\rangle \equiv b^+(\bar{r}, \bar{p})|0\rangle$

$$\begin{cases} \psi_\alpha(x)|p,r\rangle \equiv \langle 0|\psi_\alpha(x)|p,r\rangle = u_\alpha(r,p)\,e^{-ipx} \\ \langle p,r|\,\bar\psi_\alpha(x) \equiv \bar u_\alpha(r,p)\,e^{ipx} \end{cases}$$





$$\begin{cases} \bar\psi_\alpha(x)|\bar p,\bar r\rangle \equiv \bar v_\alpha(\bar r,\bar p)\,e^{-ipx} \\ \langle \bar p,\bar r|\,\psi_\alpha(x) \equiv v_\alpha(\bar r,\bar p)\,e^{ipx} \end{cases}$$



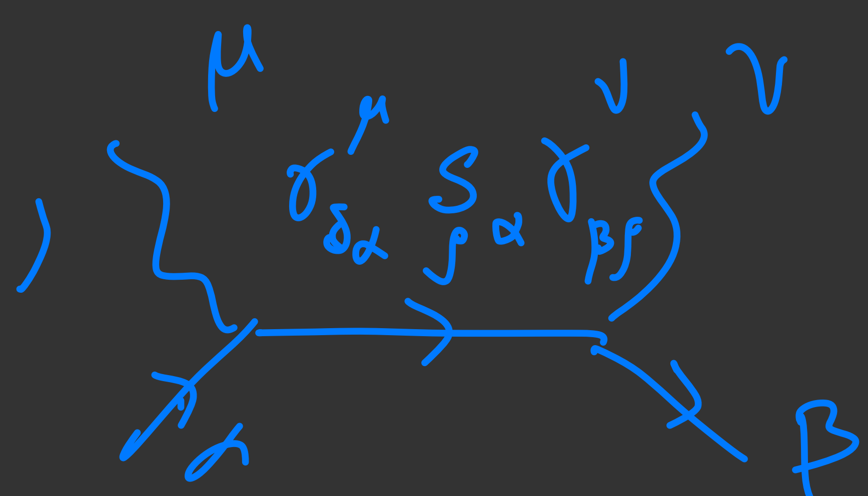
Vertex

$$\Lambda^{\mu}{}_{\nu}\,(\Lambda_D)_{\alpha\alpha'}\,(\Lambda_D^{-1})_{\beta\beta'}\,\gamma^{\nu}_{\beta'\alpha'} = \gamma^{\mu}_{\beta\alpha}$$

$$\mathcal{L}_{int} = -e A_{\mu} \bar{\psi} \gamma^{\mu} \psi \equiv -e \gamma^{\mu}_{\beta\alpha} A_{\mu} \bar{\psi}_{\beta} \psi_{\alpha}$$

$$i\mathcal{L}_{int} = -ie \gamma^{\mu}_{\beta\alpha} \underbrace{A_{\mu} \bar{\psi}_{\beta} \psi_{\alpha}}$$

$$= -ie \gamma^{\mu}_{\beta\alpha}$$



γ

- $|in\rangle = |\underbrace{(k_1\,\varepsilon_1)\cdots(k_n\,\varepsilon_n)}_{\text{photons}};\underbrace{(p_1\,r_1)\cdots(p_\ell\,r_\ell)}_{\text{electrons}};\underbrace{(\bar{p}_1\,\bar{r}_1)\cdots(\bar{p}_m\,\bar{r}_m)}_{\text{positrons}}\rangle$

- $\langle out|$ $\equiv$ same story

$$S = \sum_n S_n$$

▲ n-th term in Dyson series

$$S_n = \frac{1}{n!}(-ie)^n \int d^4x_1 \cdots d^4x_n \, T\left(A_{\mu_1}\bar{\psi}\gamma^{\mu_1}\psi(x_1)\cdots\cdots A_{\mu_n}\bar{\psi}\gamma^{\mu_n}\psi(x_n)\right)$$

$\Longrightarrow$ matrix element

$$\frac{1}{n!}(-ie)^n \int d^4x_1 \cdots d^4x_n \, \langle out|T(A\bar{\psi}\gamma\psi \;\cdots\; A\bar{\psi}\gamma\psi)|in\rangle$$

Feynman rules in momentum space

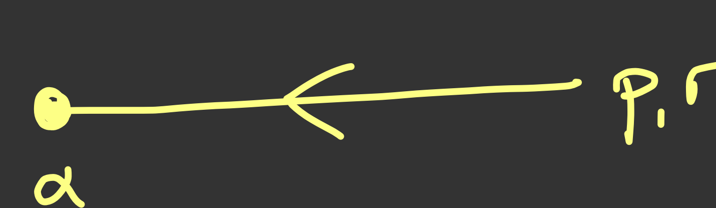
Ext lines

in-photons 

$\varepsilon_\mu(p)$

out-photons 

$\varepsilon^*_\mu(p)$

in-electrons 

$u_\alpha(p, r)$

out-electrons 

$\bar{u}_\alpha(p, r)$

in-positron 

$\bar{v}_\alpha(\bar{p}, \bar{r})$

out-position 

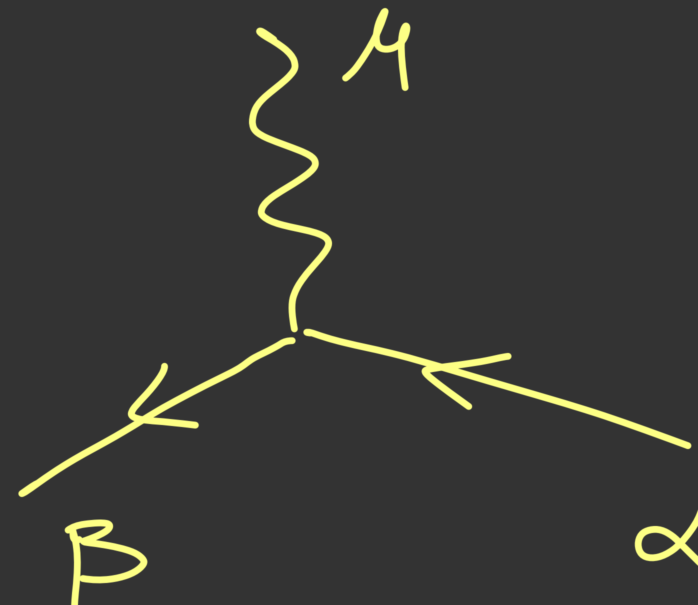
$v_\alpha(\bar{p}, \bar{r})$

Int lines

$$= -\frac{i\eta_{\mu\nu}}{p^2+i\varepsilon}$$

$$= \frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\varepsilon}$$

Vertices



$$-ie\gamma^{\mu}_{\beta\alpha}$$

- impose momentum conservation at each vertex

- integrate over residual loop momenta

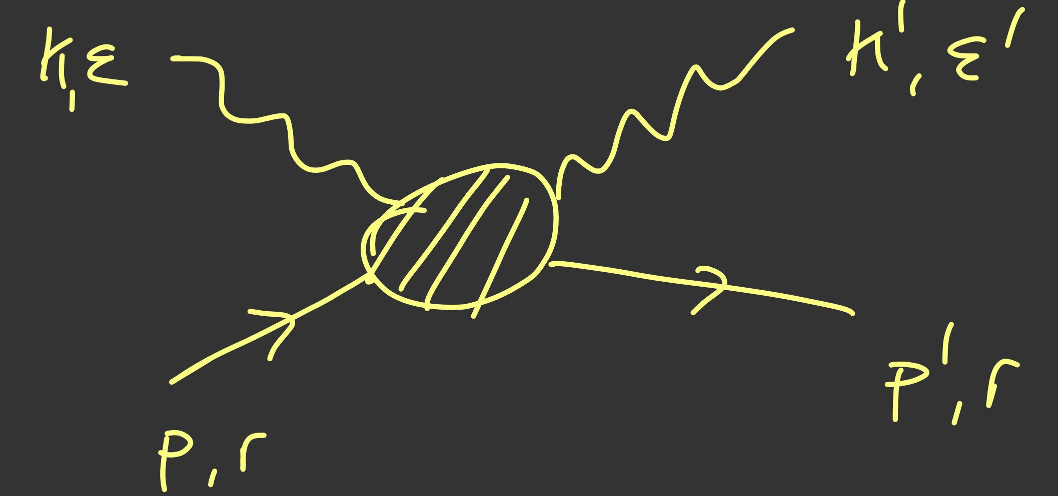
$$\int \prod_{i=1}^{N_p - N_v + 1} \frac{d^4 k_i}{(2\pi)^4} \qquad N_L \equiv N_p - N_v + 1$$

- Factor out $(2\pi)^4 \delta^4(P_f - P_i)$

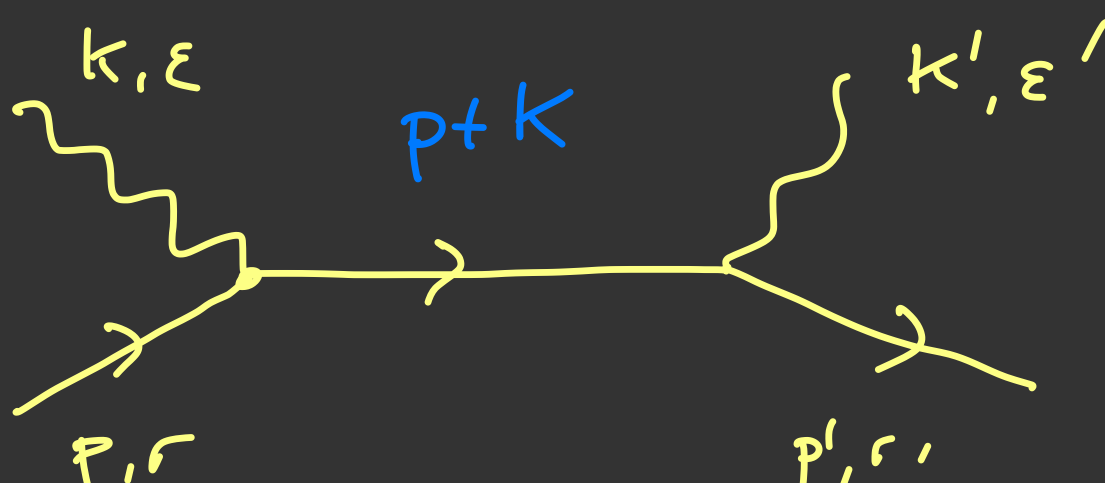
- pay attention to $(-1)^{2\pi}$ from anti-commutation of $\psi, \bar{\psi}$

Compton Scattering

$$\langle (k',\varepsilon'),(p',r')|\, S \,|(k,\varepsilon)(p,r)\rangle \;=$$



- Easy to draw lowest order diagrams



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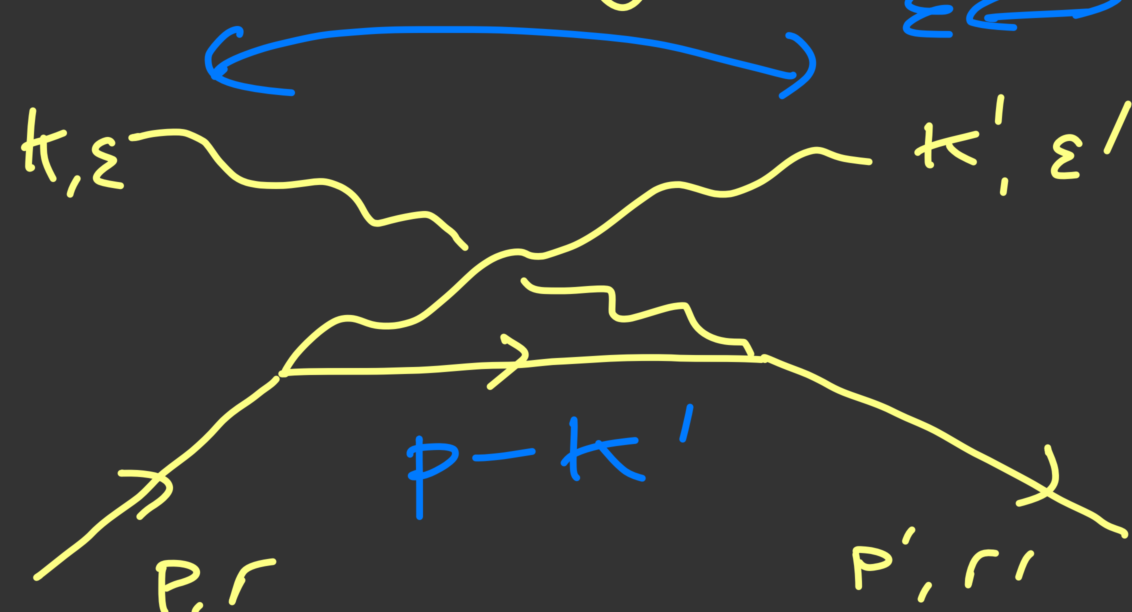
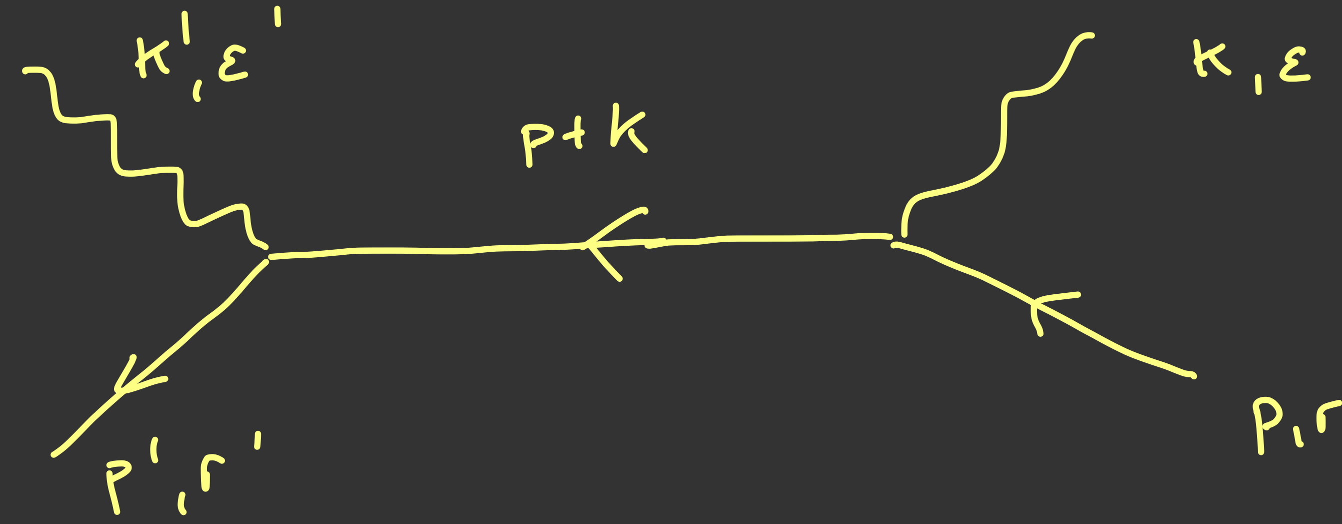


$\varepsilon \longleftrightarrow \varepsilon'^{*}$

$k \to -k'$

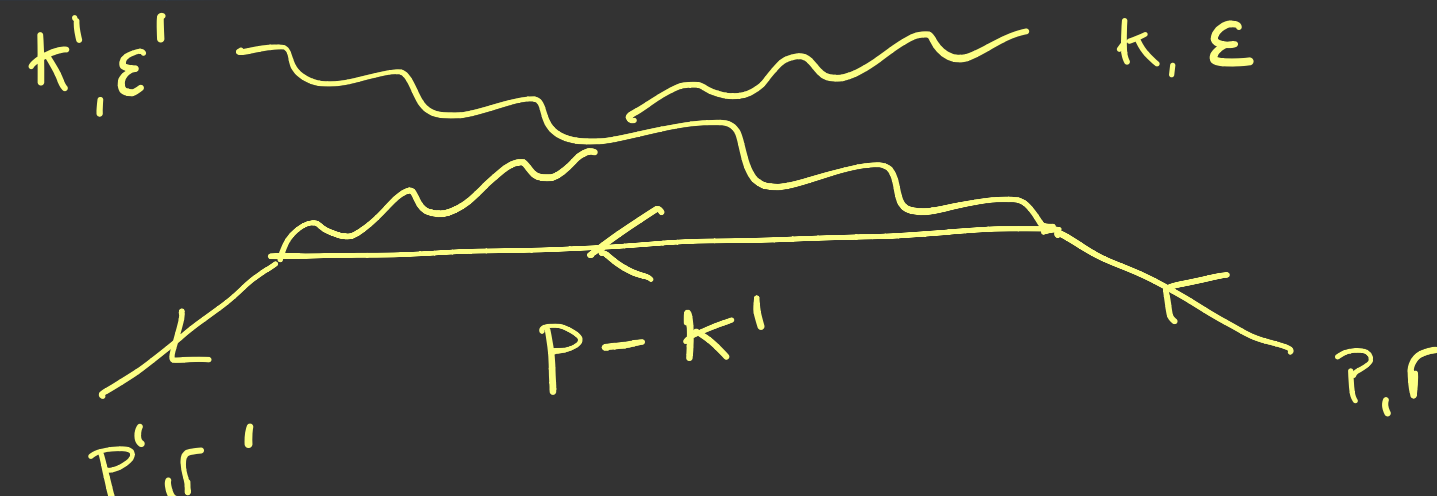
$(\varepsilon, k) \longleftrightarrow (\varepsilon'^{*}, -k')$

I)



$$\varepsilon'^{*}_{\nu}\ \bar{u}(p',r')(-ie\gamma^{\nu})\frac{i}{\not{p}+\not{k}-m}(-ie\gamma^{\mu})\,u(p,r)\,\varepsilon_{\mu}$$

II)



$$\varepsilon_{\nu}\ \bar{u}(p',r')(-ie\gamma^{\nu})\frac{i}{\not{p}-\not{k}'-m}(-ie\gamma^{\mu})\,u(p,r)\ \varepsilon'^{*}_{\mu}$$

⚠ From scratch

$$\frac{(-ie)^2}{2}\int d^4x\, d^4y\, \langle (k',\varepsilon'),(p',r') |\, T A_\nu(x)\bar\psi\gamma^\nu\psi(x)\; A_\mu(y)\bar\psi\gamma^\mu\psi(y)\, | (k,\varepsilon),(p,r)\rangle$$

I) $\langle (k',\varepsilon'),(p',r') |\, T\big(A_\nu(x)\bar\psi\gamma^\nu\psi(x)\; A_\mu(y)\bar\psi\gamma^\mu\psi(y)\big)\, | (k,\varepsilon)(p,r)\rangle$

$$\frac{(-ie)^2}{\cancel{2}}\times\cancel{2}\qquad \varepsilon'^*_\nu\, \bar u(p',r')\gamma^\nu \cdot \frac{i}{\not p+\not k-m}\cdot\gamma^\mu u(p,r)\,\varepsilon_\mu$$

same as I of previous page

$$\int d^4x\, d^4y \, \langle (k', \varepsilon'), (p', r') | \, T \left(A_\nu(x) \bar{\psi} \gamma^\nu \psi(x) \; A_\mu(y) \bar{\psi}(y) \gamma^\mu \psi(y) \right) | (k, \varepsilon)(p, r) \rangle$$

$$\int d^4x\, d^4y \, \frac{d^4 q}{(2\pi)^4} \, e^{ik'x} \varepsilon'^*_\nu \, e^{ip'x} \, \bar{u}(p', r') \gamma^\nu \frac{i e^{-iq(x-y)}}{\not{q} - m + i\varepsilon} \gamma^\mu u(p, r) e^{-ipy} \varepsilon_\mu e^{-iky}$$

$$\left.\begin{aligned} d^4x &\to k' + p' - q = 0 \\ d^4y &\to q - p - k = 0 \end{aligned}\right\} \quad q = p + k$$

$$k' + p' = k + p \Rightarrow (2\pi)^4 \delta^4(p_f - p_i)$$

$$(-e^2)\ \varepsilon'^{*}_{\nu}\,\bar{u}(p',r')\,\gamma^{\nu}\,\frac{i}{\not{p}+\not{k}-m}\,\gamma^{\mu}\,u(p,r)\,\varepsilon_{\mu}$$

precisely $\mathcal{I}$

$\frac{1}{2}\times 2 \longrightarrow x \leftrightarrows y$

$$\mathrm{II})\quad \langle (k',\varepsilon'),(p',r')|\, T\big(A_\nu(x)\,\bar\psi\,\gamma^\nu\psi(x)\;\; A_\mu(y)\,\bar\psi\,\gamma^\mu\psi(x)\big)\,|(k,\varepsilon)(p,r)\rangle$$

$$\frac{(-e^2)}{\cancel{2}}\times\cancel{2}\;\;\varepsilon'^{\,*}_{\mu}\;\bar u(p',r')\,\gamma^\nu\,\frac{i}{\not p-\not k'-m}\,\gamma^\mu\,u(p,r)\,\varepsilon^\nu$$

precisely II of previously

$$i\mathcal{M} = \mathrm{I} + \mathrm{II} =$$

$$= -ie^2 \varepsilon'^{*}_{\mu} \varepsilon_{\nu} \left[\bar{u} \left(\gamma^{\mu} \frac{1}{\not{p} + \not{k} - m} \gamma^{\nu} + \gamma^{\nu} \frac{1}{\not{p} - \not{k}' - m} \gamma^{\mu} \right) u \right]$$

- $\mathcal{M} \equiv$ manifestly Lorentz invariant
- two quantum interfering contributions... and not by chance

⚠ Gauge invariance

- recall properties of photon Hilbert space

1) $|k, \varepsilon\rangle \equiv \varepsilon_\mu a^{\mu\,+}(k)|0\rangle$ with $k^\mu \varepsilon_\mu = 0$ $\left(\langle \partial_\mu A^\mu \rangle = 0\right)$

2) relation of equivalence $\varepsilon_\mu(k) \sim \varepsilon_\mu(k) + b\, k_\mu$

1) + 2) $\Rightarrow$ only two physical polarizations

- Observables should respect $\varepsilon_\mu \sim \varepsilon_\mu + b\, k_\mu$ equivalence

$\Rightarrow$ b should not affect physical quantities

- S-matrix must be unaffected by $\varepsilon_\mu \to \varepsilon_\mu + b\kappa_\mu$

$$\mathcal{M}(\varepsilon, \varepsilon') = \mathcal{M}(\varepsilon + bk, \varepsilon' + b'k') = (\varepsilon_\mu + bk_\mu)(\varepsilon'^*_\nu + b'k'_\nu) \times \mathcal{M}^{\mu\nu}$$

- our result $\mathcal{M} = \varepsilon'^*_\mu \varepsilon_\nu \mathcal{M}^{\mu\nu}$

- $k'_\mu \mathcal{M}^{\mu\nu} = k_\nu \mathcal{M}^{\mu\nu} = 0$ would be sufficient

- $$\mathcal{M}^{\mu\nu} = -e^2 \bar{u}(p', r') \left[\gamma^\mu \frac{1}{\not{p} + \not{k} - m} \gamma^\nu + \gamma^\nu \frac{1}{\not{p} - \not{k}' - m} \gamma^\mu \right] u(p, r)$$

$$p + k = p' + k' \Rightarrow p - k' = p' - k$$

$$\Downarrow$$

$$\not{p}' - \not{k} - m$$

$$k_\nu \mathcal{M}^{\mu\nu} = -e^2\,\bar{u}\left(\gamma^\mu \frac{1}{\not{p}+\not{k}-m}\not{k} + \not{k}\frac{1}{\not{p}'-\not{k}-m}\gamma^\mu\right)u$$

let us use $(\not{p}-m)u=0$ $\quad \bar{u}(\not{p}'-m)=0$

$$= -e^2\,\bar{u}\left[\gamma^\mu \frac{1}{\not{p}+\not{k}-m}(\not{p}+\not{k}-m-\not{p}+m) + (-\not{p}'+\not{k}+m+\not{p}'-m)\frac{1}{\not{p}'-\not{k}-m}\gamma^\mu\right]u$$

$$= -e^2\,\bar{u}\left[\gamma^\mu - \gamma^\mu\right]u = 0$$

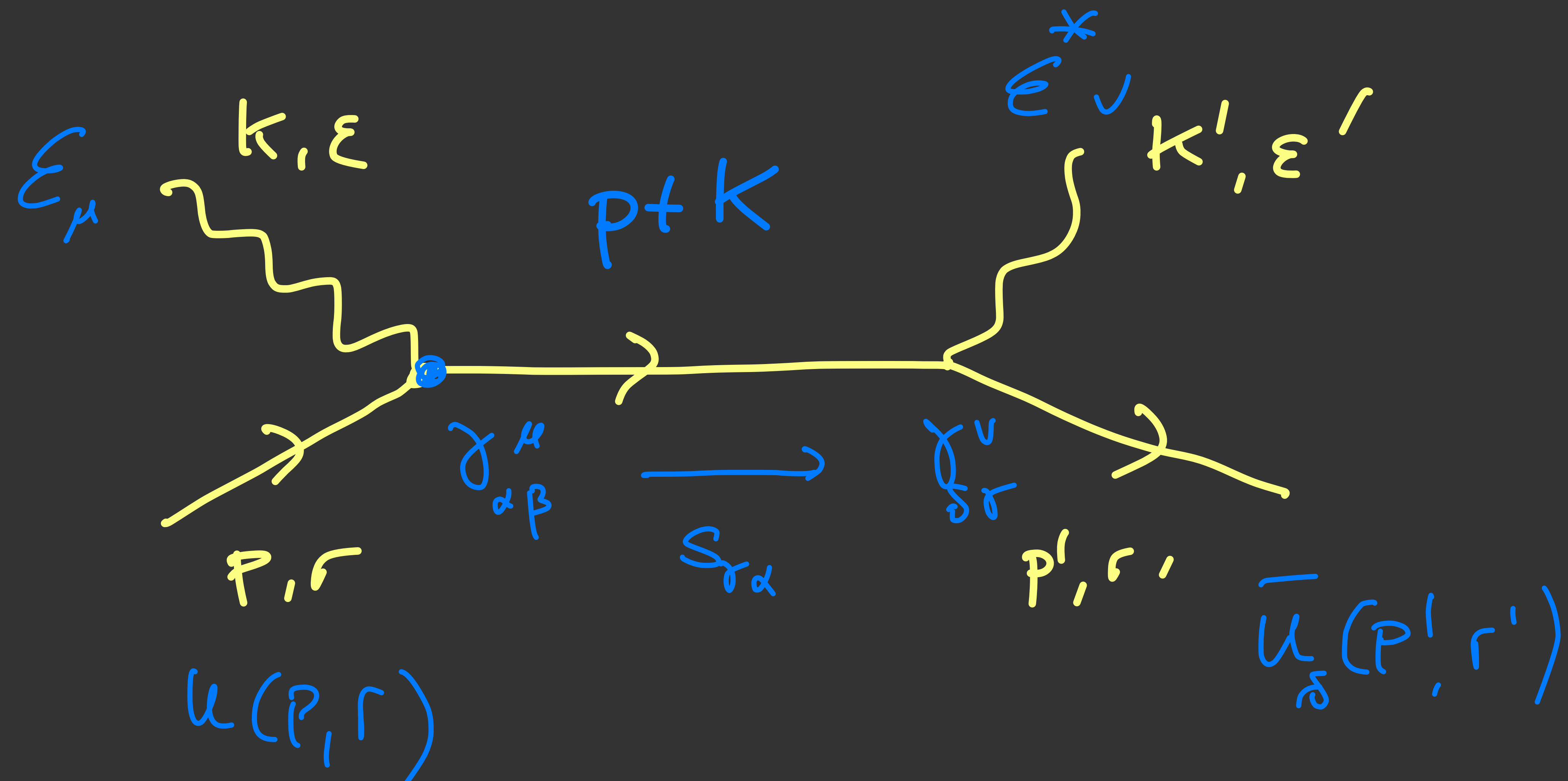
- This is the simplest instance of a result that holds $\forall$ number of external photons to all orders in perturbation theory

$\equiv$ Ward Identity

- quantum manifestation of $\partial_\mu J^\mu = 0$

$$\varepsilon_\mu \to \varepsilon_\mu + 6 K_\mu \iff A_\mu \to A_\mu + \partial_\mu \alpha$$

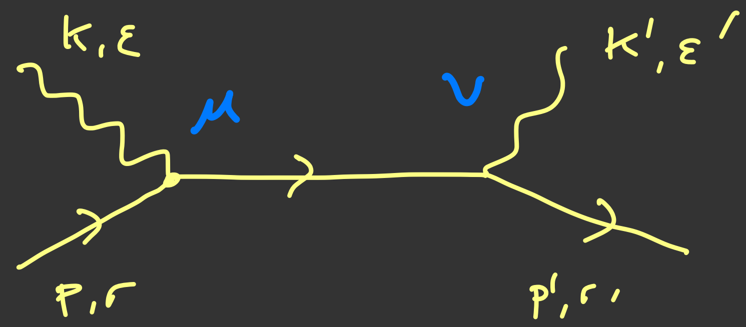
$$\int \mathcal{L}_I = -e\int A_\mu J^\mu \longrightarrow -e\int A_\mu J^\mu - e\int \alpha \underbrace{\partial_\mu J^\mu}_{\to 0}$$

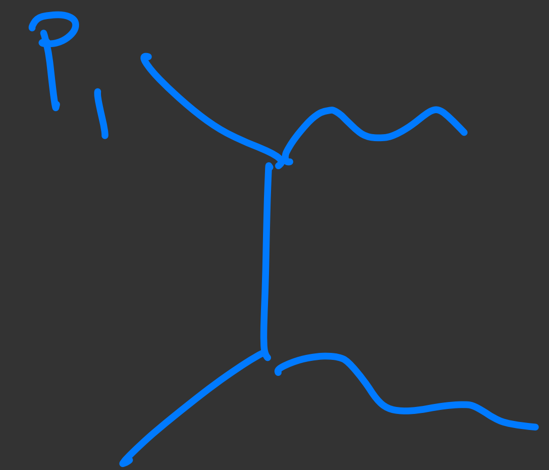


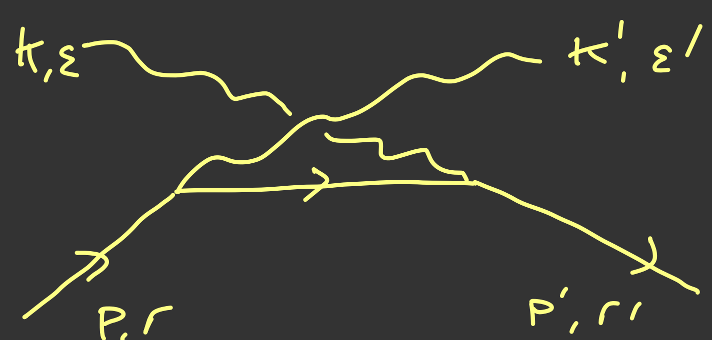
$$\varepsilon^*_\nu \, \bar{u}_\delta(p',r') \gamma^\nu_{\delta\gamma} \frac{i(\not p + \not k - m)_{\gamma\alpha}}{(p+k)^2 - m^2} \gamma^\mu_{\alpha\beta} \, u_\beta(p,r) \, \varepsilon_\mu(k)$$

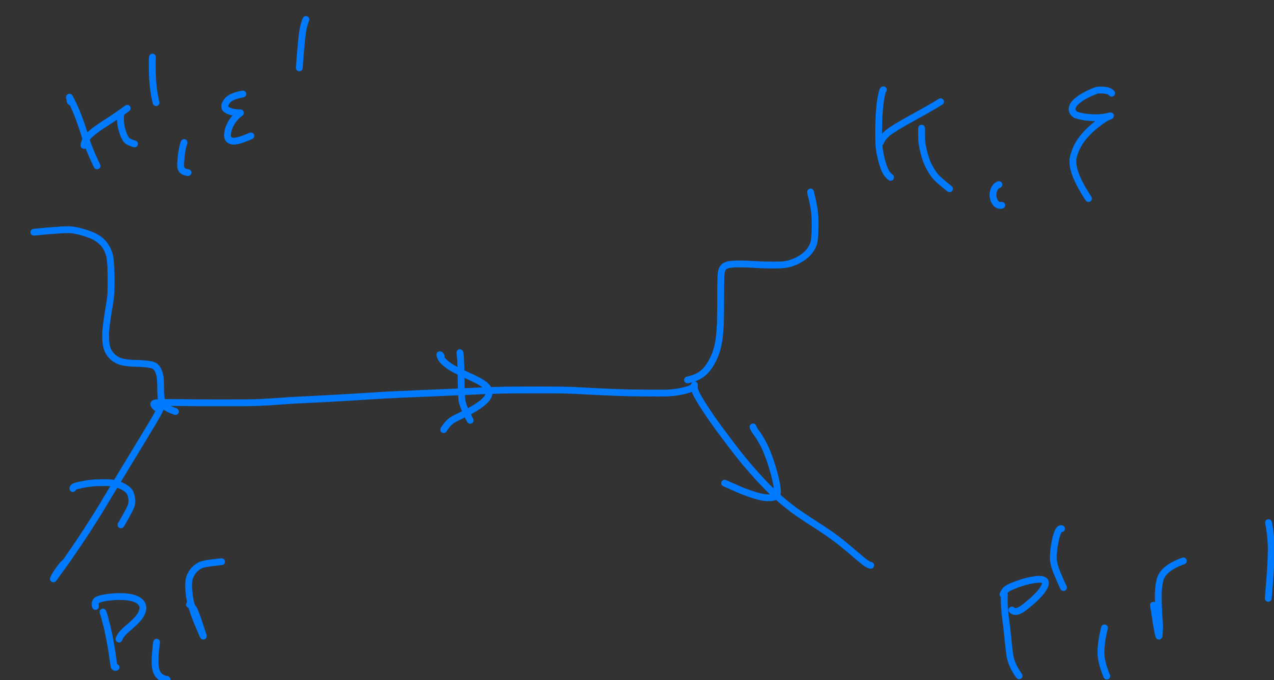
$$(-ie)^2 \varepsilon'^{*}_{\nu} \, \bar{u}(p',r') \gamma^{\nu} \frac{i(\not{p} + \not{k} + m)}{(p+k)^2 - m^2} \gamma^{\mu} \, u(p,r) \, \varepsilon_{\mu}$$

$$(-ie)^2 \varepsilon_{\nu} \, \bar{u}(p',r') \gamma^{\nu} \frac{i(\not{p} - \not{k}' + m)}{(p-k')^2 - m^2} \gamma^{\mu} \, u(p,r) \, \varepsilon'^{*}_{\mu}$$


k, ε
μ
ν
k', ε'
p, r
p', r'


p,

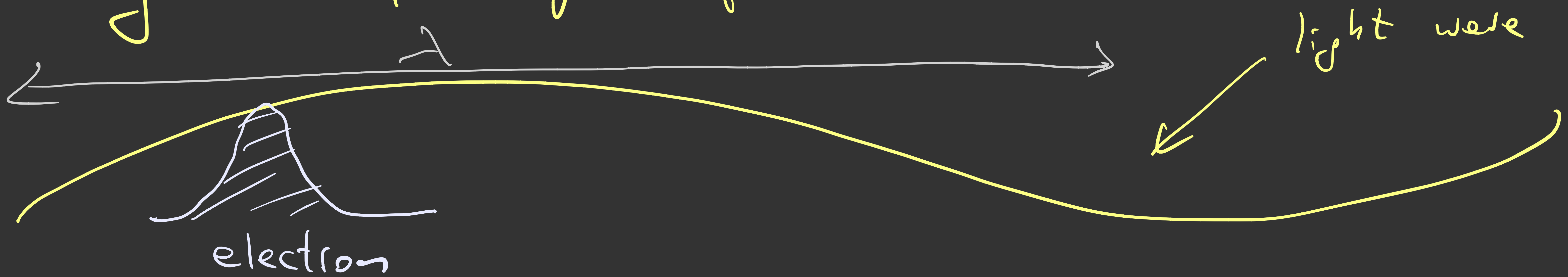

k, ε
k', ε'
p, r
p', r'


k', ε'
k, ε
p, r
p', r'

Classical vs Relativistic vs Quantum in light/matter scattering

Let us study under what condition this process can be described by classical physics.

One condition is that the electron be localized in a wave packet of size $\Delta x \ll \lambda \equiv \frac{1}{k_\gamma}$ during the passage of the EM wave



- $\Delta x \, \Delta k_e \sim 1$ + $\Delta x \ll \lambda$

$$\Longrightarrow \quad \boxed{\Delta k_e \gg k_\gamma}$$

- however this property of localization must be maintained at least for a time $\frac{1}{\omega_\gamma} \equiv \frac{1}{k_\gamma}$

A very narrow wave packet will diffuse and spread very fast. If $\Delta k_e \gtrsim m_e$ the spread velocity is $\sim 1 \Longrightarrow$ over a time $\frac{1}{\omega}$ it will spread over the whole wavelength

$\Rightarrow$ it necessarily must be $\boxed{\Delta K_e \ll m_e}$

In the end the two conditions become

$$\boxed{K_\gamma \ll \Delta K_e \ll m_e} \quad **$$

$\equiv$ non-relativistic electron scattering with very soft photon .... we could stop here

.... however $\Rightarrow$

Are these condition sufficient?
In principle not, given we should also ask the motion of electron wave packet to be semiclassical. The request of semiclassicality corresponds to asking the relevant action to be $\gg 1$. Reasonably this action should be the action corresponding to one full swing of the electron at the passage of the wave. $\Longrightarrow$

$$S \sim \int \frac{p_e^2}{2m_e} dt \sim \frac{e^2 E^2}{m_e \omega^3}$$

$$p_e \sim \frac{eE}{\omega}$$

$E =$ electric field

Need

$$\boxed{\frac{e^2 E^2}{m_e \omega^3} \gg 1}$$

can check this coincides with WKB condition from QM 3

Now $\quad \dfrac{e^2}{m_e} = r_e$

$$E^2 = \underbrace{n_\gamma}_{\text{photon density}} \cdot \omega$$

$\omega = \dfrac{1}{\lambda}$ in natural units

$$\Longrightarrow \quad \boxed{n_\gamma \cdot \lambda^2 \cdot r_e \gg 1}$$

corresponding to some "large" photon density, as expected

.... however, it turns out, by the explicit computation, the result does not depend on the photon number density (at least at lowest order) and so the classical result holds simply when ** is satisfied.

$$\Longrightarrow \quad \sigma_{\text{compton}}\Big|_{\omega_\gamma \to 0} \equiv \sigma_{\text{thomson}}$$