# Exam

## Quantum Field Theory I

**Exercise 1** **Disclaimer**: In the solution, for pedagogical reasons, we have presented almost every step of the computations (see for instance eq. 5)). No need to be that detailed at the exam, as long as what you claim is correct ☺.

In this exercise, we use greek letters ($\alpha, \beta ...$) for spinorial indices, the letters $i, j, k, ...$ for the indices of the triplet representation of isospin SU(2) and letters $a, b, c, ...$ for the indices doublet representation of isospin SU(2).

There are various ways to deal with the indices in the case when you have objects like the fermions $\psi_{a\alpha}$ one Lorentz, $\alpha$, and one isospin, $a$. In class we basically always avoided writing the Lorentz indices for spinors, though in part of section 5.2.3 of lecture notes we show how to set up a consistent convention for dealing with explicit spinor indices. So we can equivalently write

$$(\epsilon\sigma^i)_{ab}A^i\ \left[\psi_a^T\hat{\epsilon}\psi_b\right] = (\epsilon\sigma^i)_{ab}A^i\ \psi_{a\beta}\hat{\epsilon}^{\beta\alpha}\psi_{b\alpha}$$

where in the left-hand side we did not display Lorentz indices and indicated the Lorentz invariant contraction of spinors by $\left[\psi_a^T\hat{\epsilon}\psi_b\right]$. We put a hat on the Lorentz $\epsilon$ to distinguish it from the Isospin $\epsilon$, just to avoid confusion.

The isospin symmetry acts on the fields as ($U^\dagger_{ab} = U^*_{ba}$ and $U^\dagger_{ab}U_{bc} = \delta_{ac}$)

$$\Psi_a \to U_{ab}\ \Psi_b\ , \qquad \Psi_a^\dagger \to \Psi_b^\dagger (U^\dagger)_{ba} \tag{1}$$

with $U$ an $SU(2)$ matrix, which can be written as $U = e^{-i\alpha^i \frac{\sigma^i}{2}}$. $\vec{A}$ transforms as a triplet ($j = 1$) so with a matrix $R(U) \in SO(3)$ that represents $U$:

$$A_i \to R_{ij}A_j\ , \qquad R_{ij} = (e^{i\alpha^a T^a})_{ij}, \quad (T_a)_{ij} = -i\epsilon_{aij} \tag{2}$$

- Notice that $A_i$ has dimension 1 and $\psi_a$ has dimension 3/2.

  The only term of dimension 2 that we can write is just the mass term $A^T A = A_i A_i$, which is already there in the free theory.

  With dimension 3, we don't have any term that respect all the symmetries.

  With dimension 4 we can write the term we already have, $\psi_a^\dagger \bar{\sigma}^\mu \partial_\mu \psi_a$, and also $(A_i A_i)^2$. Another term we can write is

  $$A_i(\epsilon\sigma_i)_{ab}[\psi_a^T\hat{\epsilon}\psi_b] \tag{3}$$

  Under Lorentz transformations, the combination $[\psi^T\hat{\epsilon}\psi]$ is invariant since, in matrix notation the spinors transform as $\psi \to D_L(\Lambda)\psi, \psi^T \to \psi^T D_L(\Lambda)^T$, so $[\psi^T\hat{\epsilon}\psi]$ transforms as

  $$D_L(\Lambda)^T\hat{\epsilon}D_L(\Lambda) = \hat{\epsilon}D_R(\Lambda)^\dagger D_L(\Lambda) = \hat{\epsilon} \tag{4}$$

  Under $SU(2)$ transformations we have that the term in eq. 3 is invariant

  $$\begin{aligned} A_i(\epsilon\sigma_i)_{ab}[\psi_a^T\hat{\epsilon}\psi_b] \to & R(U)_{ij}A_j(U^T\epsilon\sigma_i U)_{de}[(\psi^T)_d\hat{\epsilon}\psi_e] = \\ &= R(U)_{ij}A_j(\epsilon U^\dagger\epsilon^{-1}\epsilon\sigma_i U)_{de}[(\psi^T)_d\hat{\epsilon}\psi_e] = \\ &= R(U)_{ij}A_j(\epsilon U^\dagger\sigma_i U)_{de}[(\psi^T)_d\hat{\epsilon}\psi_e] = \\ &= R(U)_{ij}A_j R(U^\dagger)_{ki}(\epsilon\sigma_k)_{de}[(\psi^T)_d\hat{\epsilon}\psi_e] = \\ &= A_j(\epsilon\sigma_j)_{de}[(\psi^T)_d\hat{\epsilon}\psi_e] \end{aligned} \tag{5}$$

  where we used $U^T = e^{-i\alpha_i\frac{\sigma_i^T}{2}} = e^{i\alpha_i\epsilon\sigma_i\epsilon^{-1}} = \epsilon e^{i\alpha_i\sigma_i}\epsilon^{-1} = \epsilon U^\dagger\epsilon^{-1}$ (remember $\sigma_i^T = -\epsilon\sigma_i\epsilon^{-1}$). We have also used that (see Set 8) $U^\dagger\sigma_i U = R(U)_{il}\sigma_l = R(U^\dagger)_{li}\sigma_l$, and $R(U)R(U^\dagger) = 1$ (and also $R(U^\dagger) = R(U)^T$, as these are just the usual orthogonal matrices of $SO(3)$).

- The infinitesimal $SU(2)$ transformations are

$$\begin{aligned} \Psi_a &\to \Psi_a - i\alpha^i \frac{(\sigma^i)_a^{\ b}}{2}\Psi_b = \Psi_a + \Delta^i_{\Psi,a}\alpha^i\,, \\ A_i &\to A_i + i\alpha^a (T^a)_i^{\ j} A_j = A_i + \alpha^a \epsilon_{aij} A_j = A_i + \Delta^a_{A,i}\alpha^a \end{aligned} \tag{6}$$

So the Noether current is

$$\begin{aligned} J^\mu_i &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^a)}\Delta^i_{\Psi,a} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^j)}\Delta^i_{A,j} = \\ &= (\psi^a)^\dagger \bar\sigma^\mu \frac{(\sigma_i)_{ab}}{2}\psi^b + (\partial^\mu A_j)\epsilon_{ijk}A_k \end{aligned} \tag{7}$$

and the charge is ($[\cdots]$ indicates Lorentz contracted spinor indices)

$$Q_i = \int d^3x\ J^0_i = \int d^3x\ \left(\frac{(\sigma_i)_{ab}}{2}[\psi^\dagger_a \psi_b] + \epsilon_{ijk}(\partial^0 A_j)A_k\right) \tag{8}$$

- In the quantized theory the fields obey the standard (anti)commutation relations

$$\begin{aligned} &\{\psi_a(x), \psi_b(y)\} = 0, \qquad [A_i(\vec x, t), A_j(\vec y, t)] = 0 \\ &\{\psi_a, \psi^\dagger_b\} = \delta_{ab}\delta^{(3)}(\vec x - \vec y), \quad [\pi_i(\vec x, t), A_j(\vec y, t)] = -i\delta_{ij}\delta^{(3)}(\vec x - \vec y), \end{aligned} \tag{9}$$

with $\pi_i(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i)} = \partial^0 A_i$.

- Putting $\psi^a = 0$ ($a = 1, 2$) we compute (for simplicity here we denote $x = (\vec x, t)$ and the same for $y$)

$$\begin{aligned} [Q_i, Q_j] &= \int d^3x \int d^3y\ \big(+\epsilon_{ilm}\epsilon_{jab}[A^l(x)\partial^0 A^m(x), A^a(y)\partial^0 A^b(y)]\big) = \\ &= \int d^3x \int d^3y\ \big(\epsilon_{ilm}\epsilon_{jab}A^l(x)[\partial^0 A^m(x), A^a(y)]\partial^0 A^b(y) + A^a(y)[A^l(x), \partial^0 A^b(y)]\partial^0 A^m(x)\big) = \\ &= \int d^3x\ \big(-iA^j(x)\partial^0 A^i(x) + iA^i(x)\partial^0 A^j(x)\big) = i\epsilon_{ijk}\int d^3x\ i\epsilon_{klm}A^l(x)\partial^0 A^m(x) = i\epsilon_{ijk}Q_k \end{aligned} \tag{10}$$

where we have used the commutation relations and the property $\epsilon_{ilm}\epsilon_{jam} = \delta_{ij}\delta_{la} - \delta_{ia}\delta_{lj}$. The $SU(2)$ algebra is correctly reproduced.

- Setting $A_i = 0$ we have

$$\begin{aligned} [Q_i, Q_j] &= \int d^3x \int d^3y\ \left(\frac{(\sigma_i)_{ab}}{2}\frac{(\sigma_j)_{cd}}{2}[\psi^{\dagger a}(x)\psi^b(x), \psi^{\dagger c}(y)\psi^d(y)]\right) = \\ &= \int d^3x \int d^3y\ \frac{(\sigma_i)_{ab}}{2}\frac{(\sigma_j)_{cd}}{2}\big(\psi^{\dagger a}(x)[\psi^b(x), \psi^{\dagger c}(y)\psi^d(y)] + [\psi^{\dagger a}(x), \psi^{\dagger c}\psi^d(y)]\psi^b(x)\big) = \\ &= \int d^3x \int d^3y\ \frac{(\sigma_i)_{ab}}{2}\frac{(\sigma_j)_{cd}}{2}\big(\psi^{\dagger a}(x)\{\psi^b(x), \psi^{\dagger c}(y)\}\psi^d(y) - \psi^{\dagger c}(y)\{\psi^{\dagger a}(x), \psi^d(y)\}\psi^b(x)\big) = \\ &= \int d^3x\ \frac{(\sigma_i)_{ab}}{2}\frac{(\sigma_j)_{bd}}{2}\psi^{\dagger a}\psi^d - \frac{(\sigma_i)_{ab}}{2}\frac{(\sigma_j)_{ca}}{2}\psi^{\dagger c}\psi^b = \int d^3x\ ([\frac{\sigma_i}{2}, \frac{\sigma_j}{2}])_{ab}\psi^{\dagger a}\psi^b = i\epsilon_{ijk}Q_k \end{aligned} \tag{11}$$

where we have used $[A, BC] = \{A, B\}C - B\{A, C\}$ and $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$. The $SU(2)$ algebra is also satisfied in this case.

**Exercise 2** In this exercise we are using the Euclidean metric such that $L^i = L_i$.

- In the first point we have to prove that $\mathbf{J}\,|\psi\rangle = \int d\Omega_1 d\Omega_2\ \{[\mathbf{L}(\mathbf{k}_1) + \mathbf{L}(\mathbf{k}_2)]f(\mathbf{k}_1, \mathbf{k}_2)\}\, a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)\,|0\rangle$, where,

$$d\Omega_k = \frac{d^3\mathbf{k}}{(2\pi)^3 2E_k}. \tag{12}$$

We can easily prove it using the commutation relations,

$$\big[a(\mathbf{k}), a^\dagger(\mathbf{p})\big] = (2\pi)^3 2E_k \delta^3(\mathbf{k} - \mathbf{p}), \tag{13}$$

and

$$J^{ij} = -i\int d\Omega_k a^\dagger(\mathbf{k})(k^i\partial^j - k_j\partial_i)a(\mathbf{k}), \tag{14}$$

and $J^k = \varepsilon^{kij}J^{ij}$. With this,

$$\begin{aligned}
\mathbf{J}\left|\psi\right\rangle &= \int d\Omega_k a^\dagger(\mathbf{k})\mathbf{L}(\mathbf{k})a(\mathbf{k})\int d\Omega_1 d\Omega_2 f(\mathbf{k}_1,\mathbf{k}_2)a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)\left|0\right\rangle \\
&= \int d\Omega_k d\Omega_1 d\Omega_2 f(\mathbf{k}_1,\mathbf{k}_2)a^\dagger(\mathbf{k})\mathbf{L}(\mathbf{k})a(\mathbf{k})a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)\left|0\right\rangle \\
&= \int d\Omega_1 d\Omega_2\ \{[\mathbf{L}(\mathbf{k}_1)+\mathbf{L}(\mathbf{k}_2)]f(\mathbf{k}_1,\mathbf{k}_2)\}\, a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)\left|0\right\rangle,
\end{aligned} \tag{15}$$

where we have used the commutation relations,

$$a(\mathbf{k})a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)\left|0\right\rangle = (2\pi)^3 2E_k\left(\delta^3(\mathbf{k}_1-\mathbf{k})a^\dagger(\mathbf{k}_2)+\delta^3(\mathbf{k}_2-\mathbf{k})a^\dagger(\mathbf{k}_1)\right)\left|0\right\rangle, \tag{16}$$

the fact that $a(\mathbf{k})\left|0\right\rangle = 0$, then we used that $\frac{\partial}{\partial k^i}\delta^3(\mathbf{k}-\mathbf{p}) = -\frac{\partial}{\partial p^i}\delta^3(\mathbf{k}-\mathbf{p})$, afterwards we evaluated the delta function and then we integrated by parts each of the terms.

- Now we go to the center of mass frame where $f(\mathbf{k}_1,\mathbf{k}_2) = \delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)g(\mathbf{k}_1)$. Notice that $(\mathbf{L}(\mathbf{k}_1)+\mathbf{L}(\mathbf{k}_2))\delta^3(\mathbf{k}_1+\mathbf{k}_2) = 0$. In this frame,

$$\left|\psi\right\rangle_{CM} = \int d\Omega_k \frac{1}{(2\pi)^3 2E_k} g(\mathbf{k})a^\dagger(\mathbf{k})a^\dagger(-\mathbf{k})\left|0\right\rangle, \tag{17}$$

With this, evaluating the $\mathbf{k}_2$ integration we have,

$$\mathbf{J}\left|\psi\right\rangle = \int d\Omega_1 \frac{1}{(2\pi)^3 2E_{k_1}}\mathbf{L}(\mathbf{k}_1)g(\mathbf{k}_1)a^\dagger(\mathbf{k}_1)a^\dagger(-\mathbf{k}_1)\left|0\right\rangle. \tag{18}$$

Notice that since $E_k = f(k^2)$, it commutes with the angular momentum.

- Now we take $\mathbf{k_1} = |\mathbf{k_1}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\phi)$. We are working in the center of mass frame with $g(\mathbf{k}_1) = (2\pi)\delta(|\mathbf{k_1}|-k)\sin(\theta)e^{-i\phi}$. We should compute $L^3(\mathbf{k}_1)g(\mathbf{k}_1)$. After an explicit computation we can see that $L^3(\mathbf{k}_1)g(\mathbf{k}_1) = -g(\mathbf{k}_1)$. This is expected because the wave function that we are given $g(\mathbf{k}_1) \propto Y_1^{-1}(\theta,\phi)$. With this, $J^3\left|\psi\right\rangle = -\left|\psi\right\rangle$.
- In order to compute the angular momentum we should of course compute $\mathbf{J}^2\left|\psi\right\rangle$. We have two ways to do it, computing $\mathbf{J}^2\left|\psi\right\rangle$ explicitly we get $\mathbf{J}^2\left|\psi\right\rangle = 2\left|\psi\right\rangle$. Then the angular momentum of the state is $l = 1$. Again this is what we expected since $g(\mathbf{k}_1) \propto Y_1^{-1}(\theta,\phi)$. We can also check that $J^-\left|\psi\right\rangle = 0$, and since $J^3\left|\psi\right\rangle = -\left|\psi\right\rangle$ with our knowledge of group theory for the angular momentum, our state must have $l = 1$.
- It is easy to guess looking at the computation we did before,

$$\mathbf{J}\left|\psi\right\rangle = \int d\Omega_1 d\Omega_2 ... d\Omega_n\ \{[\mathbf{L}(\mathbf{k}_1)+\mathbf{L}(\mathbf{k}_2)+...+\mathbf{L}(\mathbf{k}_n)]f(\mathbf{k}_1,\mathbf{k}_2,...,\mathbf{k}_n)\}\, a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)...a^\dagger(\mathbf{k}_n)\left|0\right\rangle \tag{19}$$

where we just used that the n-particle state has the form:

$$\left|\psi_n\right\rangle = \int d\Omega_1 d\Omega_2 ... d\Omega_n\, f(\mathbf{k}_1,\mathbf{k}_2,...,\mathbf{k}_n)a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)...a^\dagger(\mathbf{k}_n)\left|0\right\rangle. \tag{20}$$

- In the lecture we saw that the angular momentum for particles with spin 1/2 reads,

$$J_k = \int d^3\vec{x}\psi^\dagger\left[\vec{x}\times(-i\vec{\nabla})+\frac{\vec{\Sigma}}{2}\right]\psi, \tag{21}$$

with

$$\vec{\Sigma} = \begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}.\end{pmatrix} \tag{22}$$

As we can see, in this case we will have to consider orbital angular momentum and spin. Also we should take into account that when acting on an n-particles state, the $a$ and $a^\dagger$ satisfy anticommutation relations. Look at section 5.5.6 (Angular momentum and boosts) of the lecture notes.