Exercises in preparation for the exam

A typical exam exercise may contain two or three questions (bullet points) of this kind.

Exercise 1: consider a Dirac field triplet ψ_i and a scalar triplet ϕ_i.

- Write the most general relativistic Lagrangian invariant under $SO(3)$ up to terms with dimension $d \leq 4$. Does it change if we require invariance under $O(3)$?

Exercise 2: Consider a real neutral scalar field. The ladder operators satisfy:

$$[a(\mathbf{k}), a^\dagger(\mathbf{p})] = (2\pi)^3 2E_k \delta^3(\mathbf{k} - \mathbf{p})\,.$$

Given the following state,

$$|\psi\rangle = \int d\phi d\theta \sin\theta \left(e^{2i\phi} \sin^2\theta \, a^\dagger(\mathbf{k}_{\theta,\phi})|0\rangle\right),$$

where $\mathbf{k}_{\theta,\phi} = |\mathbf{k}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\phi)$,

- show that

$$J^3|\psi\rangle = 2|\psi\rangle \qquad \text{and} \qquad J^i J^i|\psi\rangle = 6|\psi\rangle\,,$$

where $J^i = \frac{1}{2}\epsilon^{ijk}J^{jk}$ is the angular momentum operator:

$$J^{ij} = -i \int d\Omega_k \left[a^\dagger(\mathbf{k}) \left(k^i \frac{\partial}{\partial k^j} - k^j \frac{\partial}{\partial k^i}\right) a(\mathbf{k})\right].$$

Hint: Notice that $e^{i\phi}\sin\theta = (k^1 + ik^2)/|\mathbf{k}|$.

Exercise 3: Given $T^i = \frac{1}{2} a^\dagger_\alpha \sigma^i_{\alpha\beta} a_\beta$ with the commutation relations

$$[a_\alpha, a^\dagger_\beta] = \delta_{\alpha\beta}, \qquad [a_\alpha, a_\beta] = 0$$

- Prove that $[T^i, T^j] = i\epsilon^{ijk}T^k$

Given $S^i = \frac{1}{2} b^\dagger_\alpha \sigma^i_{\alpha\beta} b_\beta$ with the anticommutation relations

$$\{b_\alpha, b^\dagger_\beta\} = \delta_{\alpha\beta}, \qquad \{b_\alpha, b_\beta\} = 0$$

- Prove that $[S^i, S^j] = i\epsilon^{ijk}S^k$

Exercise 4: Consider a $SU(2)$ scalar doublet ϕ_α ($\alpha = 1, 2$) with Lagrangian

$$\mathcal{L} = \partial_\mu \phi^\dagger_\alpha \partial^\mu \phi_\alpha - V\left(\phi^\dagger_\alpha \phi_\alpha\right) .$$

The $SU(2)$ charges are given by

$$Q_k = -i \int d^3x \left[\left(\partial_0 \phi^\dagger_\alpha\right) \frac{\sigma^k_{\alpha\beta}}{2} \phi_\beta - \phi^\dagger_\alpha \frac{\sigma^k_{\alpha\beta}}{2} \partial_0 \phi_\beta \right] , \qquad k = 1, 2, 3 .$$

- Compute the commutator $[Q_i, \phi_\alpha]$.
- *Bonus question:* use the Jacobi identities to compute $[Q_i, Q_j]$.

Exercise 5: Consider four different scalars ϕ_1, ϕ_2, ϕ_3 and ϕ_4.

- Prove that $\varepsilon^{\mu\nu\rho\sigma} \partial_\mu \phi_1 \partial_\nu \phi_2 \partial_\rho \phi_3 \partial_\sigma \phi_4$ is a total derivative.
- Write a Lorentz invariant term involving a the Levi-civita tensor $\varepsilon_{\mu\nu\rho\sigma}$ which is not a total derivative.

Exercise 6: Consider the following Lagrangian for a Dirac fermion ψ:

$$\mathcal{L} = iZ\bar{\psi}\gamma^\mu \partial_\mu \psi - M\bar{\psi}\psi - i\tilde{M}\bar{\psi}\gamma_5\psi.$$

- Find the equations of motion. What is the mass of the Dirac particle?
- Prove that the above Lagrangian can be recast in the standard Dirac form via a field redefintion of the kind $\psi \to e^{\alpha + i\beta\gamma_5}\psi$ with $\alpha, \beta \in \mathbb{R}$.

Exercise 7: consider the following Lagrangian for a scalar field ϕ:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \lambda(\partial\phi)^2 \Box\phi$$

- What is the dimensionality of λ?
- Find the equations of motion. Can you identify the symmetries of the equations of motion?

Exercise 8: consider two scalars ψ_I and ϕ_I, $I = 1, 2, 3$ transforming as triplets under an internal $SU(2)$, called Isospin.

- Decompose the product $\psi_I \phi_J$ into irreducible representations of $SU(2)$.
- Given a third scalar χ_{IJ}, what constraint should it satisfy for it to correspond to Isospin 2?
- Write the most general Lagrangian for ϕ_I, ψ_I and χ_{IJ} (satisfying the above constraints) involving only terms with dimension $d \leq 4$.

Exercise 9: Consider N left-handed fermions ψ_L^a and N right-handed fermions ψ_R^a, $a = 1, \ldots, N$ with Majorana and Dirac mass matrix

$$\begin{aligned}\mathcal{L} &= i(\psi_L^a)^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L^a + i(\psi_R^a)^\dagger \sigma^\mu \partial_\mu \psi_R^a \\ &- \left[(\psi_L^a)^\dagger\, m_{ab} \psi_R^b + h.c.\right] - \frac{1}{2}\left[(\psi_R^a)^T i\sigma_2 M_{ab} \psi_R^b + h.c.\right]\end{aligned}$$

- Show that the Lagrangian is Lorentz invariant
- For the last term to be non-vanishing, should M_{ab} be symmetric or anti-symmetric?
- Find the equations of motion. What happens if $m_{ab} = 0$?

Remark: when deriving the the equations of motion you should consider the fields and their variation as anticommuting variables. For instance given two anti commuting variables χ_1 and χ_2 we have $\delta(\chi_1\chi_2) = \delta\chi_1\chi_2 + \chi_1\delta\chi_2 = \delta\chi_1\chi_2 - \delta\chi_2\chi_1$.

Exercise 10: Given two scalar fields H_α and ϕ_α both transforming as doublet under an internal $SU(2)$, prove that

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  - $H_\alpha^* H_\alpha \equiv H^\dagger H$
  - $\phi_\alpha^* \phi_\alpha \equiv \phi^\dagger \phi$
  - $H_\alpha^* \phi_\alpha \equiv H^\dagger \phi$
  - $H_\alpha \epsilon_{\alpha\beta} \phi_\beta \equiv H^T \epsilon \phi$ $\qquad \epsilon = i\sigma_2$

  are $SU(2)$ singlets,

-
  - $H^\dagger \sigma^i H$
  - $H^T \epsilon \sigma^i H$

  are $SU(2)$ triplets,

- finally decompose explicitly into irreducible representations the following $SU(2)$ tensors
  - $(H^\dagger \sigma^i H)(H^\dagger \sigma^j H) = \mathbf{2} \oplus \mathbf{0}$
  - $(H^\dagger \sigma^i H)(\phi^\dagger \sigma^j \phi) = \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{0}$

Exercise 11: Given a Dirac spinor field ψ,

- study the transformation properties of the following bilinears

  $$\bar{\psi}\psi\,, \qquad \bar{\psi}\gamma^\mu\psi\,, \qquad \bar{\psi}\gamma^\mu\gamma^\nu\psi$$

  under the field tranformation $\psi \to e^{i\alpha + i\beta\gamma_5}\psi$, with $\alpha, \beta\ \in \mathbb{R}$.

Solutions

Exercise 1

- To build $SO(3)$ singlet we use the two invariant tensors δ_{ij} and ϵ_{ijk}. Other limitations in the construction are the Lorentz invariance of the Lagrangian and the request $d \leq 4$. We find

  $$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i - \frac{1}{2}M^2\phi^i\phi^i + i\bar{\psi}^i\partial\!\!\!/\psi^i - m\bar{\psi}^i\psi^i + ig\epsilon_{ijk}\bar{\psi}^i\psi^j\phi^k + \frac{\lambda}{4}(\phi^i\phi^i)(\phi^j\phi^j)\,,$$

  where the factors in front of the operators are conventional and the i in the fifth term is such that g is a real number (check it).

  The fifth term is invariant under $SO(3)$ transformations but not under $O(3)$ transformations, since the ϵ tensor would eventually give a factor equal to the determinant of the transformation. So the theory is invariant under $O(3)$ only if $g = 0$.

Exercise 2

- It is useful to notice immediately that

  $$L^{ij}(\mathbf{k})f(|\mathbf{k}|) = 0\,, \quad L^{ij}(\mathbf{k}) = -i\left(k^i\frac{\partial}{\partial k^j} - k^j\frac{\partial}{\partial k^i}\right)\,.$$

  Indeed $L^{ij}(\mathbf{k})$ is the differential operator generating infinitesimal rotations and an arbitrary function of $|\mathbf{k}|$ is invariant under rotations. This result corresponds to the well known fact that when working in polar coordinates the differential operator $\hat{L}^{ij}$ only depends on the angles θ, ϕ and their derivatives. Consequently, in order to integrate by parts when considering the action of $\hat{L}^{ij}$, it is sufficient to just integrate over the angles.
  Consider now a state

  $$|\Psi\rangle = \int d\cos\theta d\phi f(\mathbf{k}_{\theta,\phi})a^\dagger(\mathbf{k}_{\theta,\phi})\,|0\rangle$$

  with $f(\mathbf{k})$ an arbitrary function and $k_{\theta,\phi}$ defined as previously. Using $a(\mathbf{p})\,|0\rangle = 0$ and the commutation relations (recalling $d\Omega_p = \frac{d^3p}{(2\pi)^3 2E_p}$), we have

  $$\begin{aligned} J^{ij}|\Psi\rangle &= \int d\cos\theta\, d\phi\, d\Omega_k \left[a^\dagger(\mathbf{p})L^{ij}(\mathbf{p})\right] f(\mathbf{k}_{\theta,\phi}) \left[a(\mathbf{p}), a^\dagger(\mathbf{k}_{\theta,\phi})\right] |0\rangle \\ &= \int d\cos\theta\, d\phi \left[L^{ij}(\mathbf{k}_{\theta,\phi})f(\mathbf{k}_{\theta,\phi})\right] a^\dagger(\mathbf{k}_{\theta,\phi})\,|0\rangle\,, \end{aligned}$$

where we also used $\frac{\partial}{\partial p^i}\delta^3(\mathbf{k}-\mathbf{p}) = -\frac{\partial}{\partial k^i}\delta^3(\mathbf{k}-\mathbf{p})$ and then we integrated by parts. In the case of interest we consider

$$f(\mathbf{k}) = e^{2i\phi}\sin^2\theta = \frac{(k^1+ik^2)^2}{|\mathbf{k}|^2}$$

and we have (defining $L^i = \frac{1}{2}\epsilon^{ijk}L^{jk}$)

$$J^i|\psi\rangle = \int d\phi d\theta \sin\theta \left[L^i(\mathbf{k}_{\theta,\phi})f(\mathbf{k}_{\theta,\phi})\right] a^\dagger(\mathbf{k}_{\theta,\phi})|0\rangle\,,$$
$$J^iJ^i|\psi\rangle = \int d\phi d\theta \sin\theta \left[L^i(\mathbf{k}_{\theta,\phi})L^i(\mathbf{k}_{\theta,\phi})f(\mathbf{k}_{\theta,\phi})\right] a^\dagger(\mathbf{k}_{\theta,\phi})|0\rangle\,,$$

Recalling that the $L(\mathbf{k})$'s act trivially on functions of $|\mathbf{k}|$, an explicit computation gives: [1]

$$L^3(\mathbf{k})f(\mathbf{k}) = 2f(\mathbf{k})\,, \qquad L^i(\mathbf{k})L^i(\mathbf{k})f(\mathbf{k}) = 6f(\mathbf{k})\,,$$

and consequently

$$J^3|\psi\rangle = 2|\psi\rangle\,, \qquad J^iJ^i|\psi\rangle = 6|\psi\rangle\,.$$

Exercise 3

- using the first result in the solution of exercise 3 of set 14, we have

$$[T^i, T^j] = \frac{1}{4}a^\dagger_\alpha\left[\sigma^i,\sigma^j\right]_{\alpha\beta}a_\beta = \frac{1}{2}a^\dagger_\alpha i\epsilon^{ijk}\sigma^k_{\alpha\beta}a_\beta = i\epsilon^{ijk}T^k.$$

- The same steps can be used to show $[S^i, S^j] = i\epsilon^{ijk}S^k$.

Exercise 4

- Remember the commutation relation

$$[\phi_\alpha(t,\mathbf{x}), \pi_\beta(t,\mathbf{y})] = [\phi_\alpha(t,\mathbf{x}), \partial_0\phi^\dagger_\beta(t,\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}\,.$$

From this we find

$$[Q_i, \phi_\alpha(t,\mathbf{x})] = -i\int d^3y[\partial_0\phi^\dagger_\beta(t,\mathbf{y}), \phi_\alpha(t,\mathbf{x})]\frac{\sigma^i_{\beta\gamma}}{2}\phi_\gamma(t,\mathbf{y}) = -\frac{\sigma^i_{\alpha\gamma}}{2}\phi_\gamma(t,\mathbf{x})\,.$$

[1] The following properties imply that the function $e^{2i\phi}\sin^2\theta$ is proportional to the $m=\ell=2$ spherical harmonic.

- To find $[Q_i, Q_j]$ consider the nested commutator $[[Q_i, Q_j], \phi_\alpha(t, \mathbf{x})]$. We then use the Jacobi identity and the result of the previous calculation to find

$$\begin{aligned} [[Q_i, Q_j], \phi_\alpha] &= -[[Q_j, \phi_\alpha], Q_i] - [[\phi_\alpha, Q_i], Q_j] \\ &= \frac{\sigma^j_{\alpha\beta}}{2}[\phi_\beta, Q_i] - \frac{\sigma^i_{\alpha\beta}}{2}[\phi_\beta, Q_j] \\ &= -\left(\frac{\sigma^i_{\alpha\beta}}{2}\frac{\sigma^j_{\beta\gamma}}{2} - \frac{\sigma^j_{\alpha\beta}}{2}\frac{\sigma^i_{\beta\gamma}}{2}\right)\phi_\gamma \\ &= -i\epsilon_{ijk}\frac{\sigma^k_{\alpha\gamma}}{2}\phi_\gamma = i\epsilon_{ijk}[Q_k, \phi_\alpha]\,, \end{aligned}$$

where we have used $[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$. We then see that the Noether charges have the same algebra of the symmetry group of the theory $SU(2)$.

Exercise 5

- due to Schwarz theorem $\epsilon^{\mu\nu\rho\sigma}\partial_\rho\partial_\sigma\phi = 0$. Then

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma}\partial_\mu\phi_1\partial_\nu\phi_2\partial_\rho\phi_3\partial_\sigma\phi_4 &= \partial_\mu\left(\epsilon^{\mu\nu\rho\sigma}\phi_1\partial_\nu\phi_2\partial_\rho\phi_3\partial_\sigma\phi_4\right) \\ &= \partial_\nu\left(\epsilon^{\mu\nu\rho\sigma}\partial_\mu\phi_1\phi_2\partial_\rho\phi_3\partial_\sigma\phi_4\right) \\ &= \partial_\rho\left(\epsilon^{\mu\nu\rho\sigma}\partial_\mu\phi_1\partial_\nu\phi_2\phi_3\partial_\sigma\phi_4\right) \\ &= \partial_\sigma\left(\epsilon^{\mu\nu\rho\sigma}\partial_\mu\phi_1\partial_\nu\phi_2\partial_\rho\phi_3\phi_4\right). \end{aligned}$$

- Notice that all terms of the kind

$$\phi_1^{n_1}\phi_2^{n_2}\phi_3^{n_3}\phi_4^{n_4}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\phi_1\partial_\nu\phi_2\partial_\rho\phi_3\partial_\sigma\phi_4 \\ \propto \epsilon^{\mu\nu\rho\sigma}\partial_\mu\phi_1^{n_1+1}\partial_\nu\phi_2^{n_2+1}\partial_\rho\phi_3^{n_3+1}\partial_\sigma\phi_4^{n_4+1}\,,$$

are total derivatives by the same steps above (to see it, it is enough to rename $\phi_i^{n_i+1} \to \phi_i$). In order to build a term which is not a total derivative, we can for instance multiply $\epsilon^{\mu\nu\rho\sigma}\partial_\mu\phi_1\partial_\nu\phi_2\partial_\rho\phi_3\partial_\sigma\phi_4$ by a scalar which is different than $\phi_1, \phi_2, \phi_3, \phi_4$. The simplest possibility is to use $\Box\phi_1$:

$$(\Box\phi_1)\left(\epsilon^{\mu\nu\rho\sigma}\partial_\mu\phi_1\partial_\nu\phi_2\partial_\rho\phi_3\partial_\sigma\phi_4\right)\,.$$

One can also build other terms, e.g.

$$\partial^\lambda\phi_1\left(\epsilon^{\mu\nu\rho\sigma}\partial_\mu\phi_1\partial_\nu\phi_2\partial_\rho\phi_3\partial_\lambda\partial_\sigma\phi_4\right)\,.$$

Exercise 6

- We find the equations of motion for ψ with the usual procedure

$$(iZ\not{\partial} - M - i\gamma^5 \tilde{M})\psi = 0\,.$$

To find the mass of ψ we multiply this equation by the complex conjugate of the operator in the equation

$$\begin{aligned}&(-iZ\not{\partial} - M + i\gamma^5 \tilde{M})(iZ\not{\partial} - M - i\gamma^5 \tilde{M})\psi \\ =&(Z^2\Box + M^2 + \tilde{M}^2)\psi = 0\,,\end{aligned}$$

where we have used the fact that $\not{\partial}\not{\partial} = \Box$ (prove it, it's a simple consequence of the Clifford algebra). We then see that ψ satisfies the Klein-Gordon equation with squared mass $m_\psi^2 = (M^2 + \tilde{M}^2)/Z^2$.

- The parameters Z, M and $\tilde{M}$ are redundant and two of them can be eliminated through a field redefinition. In particular we can first rescale the field

$$\psi \to \psi/\sqrt{Z}$$

so that the Lagrangian becomes

$$\mathcal{L} = i\bar{\psi}\not{\partial}\psi - \frac{M}{Z}\bar{\psi}\psi - i\frac{\tilde{M}}{Z}\bar{\psi}\gamma^5\psi\,.$$

To see how to further eliminate one paramegter from the mass term it is easier to work with the left- and right-handed components of ψ

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}\,.$$

In this notation the mass terms become

$$-(M + i\tilde{M})\bar{\psi}_L\psi_R - (M - i\tilde{M})\bar{\psi}_R\psi_L = -me^{i\theta}\bar{\psi}_L\psi_R - me^{-i\theta}\bar{\psi}_R\psi_L$$

with

$$m \equiv \sqrt{M^2 + \tilde{M}^2} \qquad \text{and} \qquad \theta \equiv \arctan(\tilde{M}/M)\,.$$

In this form we see that we can eliminate the θ phase with a field redefinition like

$$\psi_L \to e^{i\theta/2}\psi_L, \qquad \psi_R \to e^{-i\theta/2}\psi_R$$

that in terms of the full Dirac spinor ψ would be written as $\psi \to e^{-i\gamma^5\theta/2}\psi$. With this redefinitions we find the usual Dirac Lagrangian where the mass of the particle is given by

$$m_\psi = \frac{\sqrt{M^2 + \tilde{M}^2}}{Z}\,,$$

consistently with what we found before.

Exercise 7

- we have $[\mathcal{L}] = 4$, $[\partial] = [\phi] = 1$, hence

$$[\mathcal{L}] = [\lambda] + 4[\partial] + 3[\phi] = 0 \quad \Longrightarrow \quad [\lambda] = -3\,.$$

- the variation of the action is

$$\begin{aligned}\delta S &= \int d^4x \left[\partial_\mu\phi\partial^\mu\delta\phi - 2\lambda\left(\partial_\mu\phi\partial^\mu\delta\phi\right)\Box\phi - \lambda\left(\partial\phi\right)^2\Box\delta\phi\right] \\ &= \int d^4x\,\delta\phi\left[-\Box\phi + 2\lambda\partial^\mu\left(\partial_\mu\phi\Box\phi\right) - \lambda\Box\left(\partial\phi\right)^2\right] \\ &= \int d^4x\,\delta\phi\left[-\Box\phi + 2\lambda\left(\Box\phi\right)^2 - 2\lambda\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi\right]\,,\end{aligned}$$

where we integrated by parts after the first line. The EOM then is

$$\frac{\delta S}{\delta\phi} = -\Box\phi + 2\lambda\left(\Box\phi\right)^2 - 2\lambda\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi = 0.$$

The EOM is manifestly invariant under Poincaré transformations. Notice also that ϕ always appears with two derivatives acting on it. Then, since $\partial_\mu\partial_\nu x^\rho = 0$, another symmetry is

$$\phi \to \phi + b^\mu x_\mu\,,$$

where b^μ is an arbitrary (constant) four-vector. [2]

Exercise 8

- We have already decomposed the product of two spin-1 representations of $SU(2)$ in Problem Set 7 Exercise 2, where we found

$$\mathbf{1}\otimes\mathbf{1} = \mathbf{0}\oplus\mathbf{1}\oplus\mathbf{2}\,,$$

where the three irreducible representations of the decomposition are respectively the trace, the antisymmetrization and the traceless symmetrization of the product. Explicitly

$$\psi_I\phi_J = \frac{1}{3}\psi_K\phi_K\delta_{IJ} + \frac{1}{2}(\psi_I\phi_J - \psi_J\phi_I) + \frac{1}{2}(\psi_I\phi_J + \psi_J\phi_I - \frac{2}{3}\psi_K\phi_K\delta_{IJ})$$

- From this we see that a two-index tensor field χ_{IJ} corresponds to an Isospin-2 representation when

$$\chi_{IJ} = \chi_{JI} \qquad \text{and} \qquad \delta_{IJ}\chi_{IJ} = 0\,.$$

[2] Under this transformation the Lagrangian is invariant only up to a total derivative.

- The most general Lagrangian containing all three fields is

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\partial_\mu\phi_I\partial^\mu\phi_I + \frac{1}{2}\partial_\mu\psi_I\partial^\mu\psi_I + \frac{1}{2}\partial_\mu\chi_{IJ}\partial^\mu\chi_{IJ} \\ &- \frac{1}{2}m_\phi^2\phi_I\phi_I - \frac{1}{2}m_\psi^2\psi_I\psi_I - \frac{1}{2}m_\chi^2\chi_{IJ}\chi_{IJ} \\ &+ g_1(\phi_I\phi_J\chi_{IJ}) + g_2(\psi_I\psi_J\chi_{IJ}) + g_3(\phi_I\psi_J\chi_{IJ}) \\ &+ \mu_1(\epsilon_{IJK}\phi_I\psi_J\chi_{KL}\phi_L) + \mu_2(\epsilon_{IJK}\phi_I\psi_J\chi_{KL}\psi_L) \\ &+ \lambda_1(\phi_I\phi_I)(\phi_J\phi_J) + \lambda_2(\psi_I\psi_I)(\psi_J\psi_J) \\ &+ \lambda_3(\phi_I\phi_I)(\psi_J\psi_J) + \lambda_4(\chi_{IJ}\chi_{IJ})(\chi_{KL}\chi_{KL}) \\ &+ \lambda_5(\chi_{IJ}\chi_{IJ})(\phi_K\phi_K) + \lambda_6(\chi_{IJ}\chi_{IJ})(\psi_K\psi_K) \\ &+ \lambda_7(\chi_{IJ}\chi_{IJ})(\phi_K\psi_K) + \lambda_8(\chi_{IJ}\chi_{JK}\chi_{KL}\chi_{LI})\,.\end{aligned}$$

Exercise 9

- A Lorentz transformation on spacetime coordinates and on the spinor fields is defined by

$$\begin{aligned} x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad & \partial_\mu = \Lambda_\mu{}^\nu \partial_\nu \qquad \Lambda^\mu{}_\rho\Lambda_\nu{}^\rho = \delta^\mu_\nu \\ \psi'_L(x') = \Lambda_L\psi_L(x) \qquad & \psi'_R(x') = \Lambda_R\psi_R(x) \qquad \Lambda_L^{-1} = \Lambda_R^\dagger \end{aligned}$$

where Λ, Λ_L and Λ_R furnish the same Lorentz transformation in respectively the $(1/2, 1/2)$, $(1/2, 0)$ and $(0, 1/2)$ representation. In line with the discussion in class we have not explicitly shown the spinor indices (an approach using the indices is outlined below following section 5.3 of the lecture notes, but it is strictly not necessary). In class we furthermore showed that

$$\Lambda_L\sigma^\mu\Lambda_L^\dagger = \sigma^\nu\Lambda^\mu_\nu \qquad \Lambda_R\bar{\sigma}^\mu\Lambda_R^\dagger = \bar{\sigma}^\nu\Lambda^\mu_\nu$$

so that from the above results we have

$$\begin{aligned} \sigma^\mu\partial'_\mu = \Lambda_L\sigma^\mu\partial_\mu\Lambda_L^\dagger \qquad & \bar{\sigma}^\mu\partial'_\mu = \Lambda_R\bar{\sigma}^\mu\partial_\mu\Lambda_R^\dagger \\ \bar{\sigma}^\mu\partial'_\mu\psi'_L(x') = \Lambda_R\bar{\sigma}^\mu\partial_\mu\psi_L(x) \qquad & \sigma^\mu\partial'_\mu\psi'_R(x') = \Lambda_L\sigma^\mu\partial_\mu\psi_R(x)\,. \end{aligned}$$

The last two equations, together with $\Lambda_L^{-1} = \Lambda_R^\dagger$, immediately imply the Lorentz invariance of the first three terms in the action. Invariance of the fourth term follows from $\Lambda_R^T\epsilon\Lambda_R = \epsilon$ which was also derived in class.

Alternatively we could follow the index notation discussed in section 5.3 of the lecture notes. A quick way to check Lorentz invariance is to write explicitly *primed* and *unprimed* indices. In the notation of chapter of 5 of the lecture notes, using $i\sigma_2 = \epsilon = -\epsilon^{-1}$, the Lagrangian is written as

$$\begin{aligned}\mathcal{L} &= i[(\psi_L^a)^*]^{A'}(\bar{\sigma}^\mu\partial_\mu)_{A'A}(\psi_L^a)^A + i[(\psi_R^a)^*]_A(\sigma^\mu\partial_\mu)^{AA'}(\psi_R^a)_{A'} \\ &- \left\{[(\psi_L^a)^*]^{A'} m_{ab}(\psi_R^b)_{A'} + h.c.\right\} - \frac{1}{2}\left[(\psi_R^a)_{A'}\epsilon^{A'B'}M_{ab}(\psi_R^b)_{B'} + h.c.\right].\end{aligned}$$

As all lower primed (unprimed) indices are contracted with upper primed (unprimed) indices, the Lagrangian is Lorentz invariant.

- Since the fields are anticommuting variables, using $\epsilon^{AB} = -\epsilon^{BA}$, we find

$$\begin{aligned}(\psi_R^a)^T i\sigma^2 M_{ab}(\psi_R^b) &= (\psi_R^a)_{A'}\epsilon^{A'B'}M_{ab}(\psi_R^b)_{B'} = -(\psi_R^b)_{B'}\epsilon^{A'B'}M_{ab}(\psi_R^a)_{A'} \\ &= (\psi_R^b)_{B'}\epsilon^{B'A'}M_{ab}(\psi_R^a)_{A'} = (\psi_R^a)_{B'}\epsilon^{B'A'}M_{ba}(\psi_R^b)_{A'} \\ &= (\psi_R^a)^T i\sigma^2 (M)_{ba}(\psi_R^b)\end{aligned}$$

which implies $M_{ab} = M_{ba}$, hence M should be symmetric.

- To derive the EOMs, it is useful to write the Lagrangian explicitly: [3]

$$\begin{aligned}\mathcal{L} &= i(\psi_L^a)^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L^a + i(\psi_R^a)^\dagger\sigma^\mu\partial_\mu\psi_R^a \\ &- \left[(\psi_L^a)^\dagger\, m_{ab}\psi_R^b + (\psi_R^a)^\dagger\, m_{ab}^\dagger\psi_L^b\right] \\ &- \frac{1}{2}\left\{(\psi_R^a)^T i\sigma_2 M_{ab}\psi_R^b - (\psi_R^a)^\dagger i\sigma_2 M_{ab}^\dagger[(\psi_R^b)^\dagger]^T\right\}.\end{aligned}$$

Then, as the Lagrangian does not contain derivatives of $(\psi_L^a)^\dagger$, one equation of motion is

$$\frac{\partial\mathcal{L}}{\partial(\psi_L^a)^\dagger} = i\bar{\sigma}^\mu\partial_\mu\psi_L^a - m_{ab}\psi_R^b = 0\,.$$

To find the other EOM, it is convenient to restore indices and vary the Lagrangian with respect to $[(\psi_R^a)^*]_A$:

$$\begin{aligned}\delta\mathcal{L} &= i[(\delta\psi_R^a)^*]_A[\sigma^\mu\partial_\mu\psi_R^a]^A - [(\delta\psi_R^a)^*]_A m_{ab}^\dagger(\psi_L^b)^A \\ &- \frac{1}{2}\left\{[(\delta\psi_R^a)^*]_A\epsilon^{AB}M_{ab}^\dagger[(\psi_R^b)^*]_B + [(\psi_R^a)^*]_A\epsilon^{AB}M_{ab}^\dagger[(\delta\psi_R^b)^*]_B\right\} \\ &= [(\delta\psi_R^a)^*]_A\left\{[i\sigma^\mu\partial_\mu\psi_R^a]^A - m_{ab}^\dagger(\psi_L^b)^A - \frac{1}{2}\epsilon^{AB}(M_{ab}^\dagger + M_{ba}^\dagger)[(\psi_R^b)^*]_B\right\}\end{aligned}$$

where we used that $(\psi_R^a)^*$ and $(\delta\psi_R^b)^*$ anti-commute and the antisymmetry of ϵ^{AB}. Assuming M to be symmetric, we get

$$\frac{\delta S}{\delta(\psi_R^a)^*} = i\sigma^\mu\partial_\mu\psi_R^a - m_{ab}^\dagger\psi_L^b + M_{ab}^\dagger i\sigma^2(\psi_R^b)^* = 0\,.$$

For $m_{ab} = 0$, the EOMs for ψ_L and ψ_R become independent.

[3] Recall that for two Grassmanian variables $(\chi_1\chi_2)^* = \chi_2^*\chi_1^*$ (see problem set 12 ex. 3).

Exercise 10

- For this exercise we need a few relations that we proved during the semester. In particular we need the fact that $\epsilon = i\sigma^2$ is an invariant tensor of $SU(2)$ (Problem Set 9 Exercise 2)

$$U\epsilon U^T = \epsilon, \qquad U \in SU(2)$$

and the fact that the fundamental representation of $SO(3)$ is the adjoint representation of $SU(2)$ (Problem Set 5 Exercise 3)

$$U^\dagger \sigma^i U = R^i_j \sigma^j \qquad R \in SO(3), .$$

- Given these relations is immediate to prove all the results of the first two point of the exercise:

$$\begin{aligned} H^\dagger H &\to H^\dagger U^\dagger U H = H^\dagger H \\ \phi^\dagger \phi &\to \phi^\dagger U^\dagger U \phi = \phi^\dagger \phi \\ H^\dagger \phi &\to H^\dagger U^\dagger U \phi = H^\dagger \phi \\ H^T \epsilon \phi &\to H^T U^T \epsilon U \phi \to H^T \epsilon \phi \end{aligned}$$

are all singlets while

$$\begin{aligned} H^\dagger \sigma^i H &\to H^\dagger U^\dagger \sigma^i U H = R^i_j H^\dagger \sigma^j H \\ H^T \epsilon \sigma^i H &\to H^T U^T \epsilon \sigma^i U H = H^T \epsilon U^\dagger \sigma^i U H = R^i_j H^T \epsilon \sigma^j H \end{aligned}$$

are triplets.

- Since they are triplet we can use the decomposition of Exercise 10 of this sheet to get

$$\begin{aligned} (H^\dagger \sigma^i H)(H^\dagger \sigma^j H) = & \frac{1}{3}(H^\dagger \sigma^k H)(H^\dagger \sigma^k H)\delta^{ij} \\ & + \frac{1}{2}\Big[(H^\dagger \sigma^i H)(H^\dagger \sigma^j H) + (H^\dagger \sigma^j H)(H^\dagger \sigma^i H) \\ & - \frac{2}{3}(H^\dagger \sigma^k H)(H^\dagger \sigma^k H)\delta^{ij}\Big] \end{aligned}$$

and

$$\begin{aligned} (H^\dagger \sigma^i H)(\phi^\dagger \sigma^j \phi) = & \frac{1}{3}(H^\dagger \sigma^k H)(\phi^\dagger \sigma^k \phi)\delta^{ij} \\ & + \frac{1}{2}\Big[(H^\dagger \sigma^i H)(\phi^\dagger \sigma^j \phi) - (H^\dagger \sigma^j H)(\phi^\dagger \sigma^i \phi)\Big] \\ & + \frac{1}{2}\Big[(H^\dagger \sigma^i H)(\phi^\dagger \sigma^j \phi) + (H^\dagger \sigma^j H)(\phi^\dagger \sigma^i \phi) \\ & - \frac{2}{3}(H^\dagger \sigma^k H)(\phi^\dagger \sigma^k \phi)\delta^{ij}\Big] . \end{aligned}$$

In the first case the antisymmetric part of course vanishes.

Exercise 11

- The thing to remember is that γ^5 anticommutes with all the gamma matrices. This means that $e^{i\alpha}\gamma^\mu = \gamma^\mu \, e^{i\alpha}$, but $e^{i\alpha\gamma^5}\gamma^\mu = \gamma^\mu \, e^{-i\alpha\gamma^5}$. We have

$$\begin{aligned} \psi &\to e^{i\alpha + i\beta\gamma^5}\psi \\ \bar{\psi} &\to \bar{\psi} e^{-i\alpha + i\beta\gamma^5} \, . \end{aligned}$$

  We will also need the following result

$$e^{i\beta\gamma_5} = \cos\beta + i\sin\beta\gamma^5$$

  that can be easily proven by Taylor expanding the exponential and using $(\gamma^5)^2 = 1$. With these results we find

$$\begin{aligned} \bar{\psi}\psi &\to \bar{\psi}e^{i2\beta\gamma^5}\psi = \cos 2\beta \, \bar{\psi}\psi + i\sin 2\beta \, \bar{\psi}\gamma^5\psi \\ \bar{\psi}\gamma^\mu\psi &\to \bar{\psi}\gamma^\mu\psi \\ \bar{\psi}\gamma^\mu\gamma^\nu\psi &\to \bar{\psi}\gamma^\mu\gamma^\nu e^{i2\beta\gamma^5}\psi = \cos 2\beta \, \bar{\psi}\gamma^\mu\gamma^\nu\psi + i\sin 2\beta \, \bar{\psi}\gamma^\mu\gamma^\nu\gamma^5\psi \end{aligned}$$