Exam

Quantum Field Theory I

Exercise 1 Consider the following Lagrangian involving three triplets of scalar fields A_i, B_i and C_i with $i = 1, 2, 3$ (we have therefore $3 \times 3 = 9$ scalar fields)

$$\begin{aligned}\mathscr{L} =& \frac{1}{2}(\partial_\mu A_i)(\partial^\mu A_i) + \frac{1}{2}(\partial_\mu B_i)(\partial^\mu B_i) + \frac{1}{2}(\partial_\mu C_i)(\partial^\mu C_i) \\ &- \frac{1}{2}m_A^2 A_i A_i - \frac{1}{2}m_B^2 B_i B_i - \frac{1}{2}m_C^2 C_i C_i - \lambda \epsilon^{ijk} A_i B_j C_k\end{aligned}$$

a) What are the dimensions of the parameters appearing in $\mathscr{L}$?

b) What are the symmetries of $\mathscr{L}$?

c) What would they be, if we set $\lambda = 0$?

d) Limiting the analysis to dimension ≤ 4, can you find more terms invariant under the symmetries of $\mathscr{L}$?

e) Find the Noether currents and charges associated with the continuous internal symmetries of $\mathscr{L}$.

f) Consider now the two additional terms

$$\Delta\mathscr{L}_1 = A_1 B_2 - A_2 B_1, \qquad \Delta\mathscr{L}_2 = C_3$$

How do the symmetries of $\mathscr{L}$ change by adding one or both of these terms?

Exercise 2

- **Theory Question:** Discuss succinctly the various examples of spinorial wave equations.

Given a bi-spinor quantum field $\psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$ satisfying the fixed time anticommutation relations

$$\{\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{y})\} = \{\psi_\alpha^\dagger(\mathbf{x}), \psi_\beta^\dagger(\mathbf{y})\} = 0\,, \qquad \{\psi_\alpha(\mathbf{x}), \psi_\beta^\dagger(\mathbf{y})\} = \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x} - \mathbf{y}),$$

and with Hamiltonian

$$H = \int d^3\mathbf{x} \left[\psi_\alpha^\dagger (i\boldsymbol{\nabla} \cdot \vec{\sigma})_{\alpha\beta}\psi_\beta + \frac{m}{2}\psi_\alpha^\dagger \psi_\beta^\dagger \epsilon^{\alpha\beta} - \frac{m}{2}\psi_\alpha \psi_\beta \epsilon^{\alpha\beta}\right]$$

with m real, prove that

a) H is hermitian,

b) the time evolution equation

$$\dot{\psi}(\mathbf{x}) \equiv i[H, \psi(\mathbf{x})]$$

coincides with the Majorana wave equation

$$i(\bar{\sigma}^\mu \partial_\mu \psi)_\alpha = m\epsilon_{\alpha\beta}\psi_\beta^\dagger. \tag{1}$$

As shown in the course, eq. (1) implies $(\partial^2 + m^2)\psi = 0$, whose general solution can thus be expanded in plane waves as ($\omega_\mathbf{p} = \sqrt{\mathbf{p}^2 + m^2}$)

$$\psi(t, \mathbf{x}) = \int d\Omega_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} \left(e^{-i\omega_\mathbf{p} t}\xi_+(\mathbf{p}) + e^{i\omega_\mathbf{p} t}\xi_-(\mathbf{p})\right)$$

c) Show that $\psi(t, \mathbf{x}) \equiv \psi(x)$ can be rewritten as ($px \equiv p^\mu x_\mu \equiv \omega_{\mathbf{p}} t - \mathbf{p} \cdot \mathbf{x}$, $p^\mu \equiv (\omega_{\mathbf{p}}, \mathbf{p})$)

$$\psi(x) = \int d\Omega_{\mathbf{p}} \left(e^{-ipx} \xi_+(\mathbf{p}) + e^{ipx} \xi_-(-\mathbf{p}) \right)$$

d) Show that eq. (1) along with its complex conjugate imply ($p \cdot \bar{\sigma} \equiv p^\mu \bar{\sigma}_\mu$, $p \cdot \sigma \equiv p^\mu \sigma_\mu$)

$$p \cdot \bar{\sigma}\, \xi_+(\mathbf{p}) = m\, \epsilon \xi_-^*(-\mathbf{p}) \,, \qquad\qquad p \cdot \sigma\, \epsilon \xi_-^*(-\mathbf{p}) = m\, \xi_+(\mathbf{p})$$

e) Do you recognize the equation satisfied by the 4-spinor $\Xi = \begin{pmatrix} \xi_+(\mathbf{p}) \\ \epsilon \xi_-^*(-\mathbf{p}) \end{pmatrix}$?

f) If so, use the results of the course to argue that the solution takes the form

$$\xi_+(\mathbf{p}) = \sqrt{p \cdot \sigma}\, \xi(\mathbf{p}) \,, \qquad\qquad \xi_-(-\mathbf{p}) = -\sqrt{p \cdot \sigma}\, \epsilon \xi(\mathbf{p})^*$$

with $\xi(\mathbf{p})$ a generic bi-spinor.