Test Exam

Quantum Field Theory I

Exercise 1: Consider a field theory with $SU(2)$ symmetry (Isospin) and the following field content: an Isospin doublet made of left-handed Weyl spinors and an Isospin triplet made of Lorentz scalars:

$$\Psi^a \equiv \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \qquad A^i \equiv \begin{pmatrix} A^1 \\ A^2 \\ A^3 \end{pmatrix}$$

(notice: ψ_1 and ψ_2 are each a left-handed bi-spinor)

- The quadratic Lagrangian is (sum over repeated indices):

$$\mathcal{L} = \frac{1}{2}\partial_\mu A^i \partial^\mu A^i - \frac{m_A^2}{2} A^i A^i + i(\psi^a)^\dagger \bar{\sigma}^\mu \partial_\mu \psi^a \tag{1}$$

  Write down the interaction terms invariant under Poincaré and Isospin SU(2) and with dimension ≤ 4.

- **Theory Question:** Discuss succintly (say half a page) the Noether theorem.
- Compute the Noether current associated to the $SU(2)$ symmetry. Show that the conserved charges are

$$Q_i = \int d^3x \, i\epsilon_{ijk} A^j \partial_0 A^k + \frac{(\sigma_i)_{ab}}{2} (\psi^a)^\dagger \psi^b \tag{2}$$

- Write the (anti)commutation relations for the quantized ψ^a and A^i?
- Setting $\psi^1 = \psi^2 = 0$, compute the commutation relations $[Q_i, Q_j]$. Do they reproduce the $SU(2)$ algebra?
- Now instead, setting $A_i = 0$ and keeping $\psi^a \neq 0$, compute again $[Q_i, Q_j]$? Is the result different?
  We recall the distributive properties of commutators:
  $[AB, C] = A[B, C] + [A, C]B$ *and* $[AB, C] = A\{B, C\} - \{A, C\}B$.

Exercise 2: Consider a generic two-particle state for the massive Klein-Gordon field

$$|\psi\rangle = \int d\Omega_1 d\Omega_2 \, f(\mathbf{k}_1, \mathbf{k}_2) a^\dagger(\mathbf{k}_1) a^\dagger(\mathbf{k}_2) |0\rangle$$

and the angular momentum operator

$$\mathbf{J} = \int d\Omega_k \, a^\dagger(\mathbf{k}) \mathbf{L}(\mathbf{k}) a(\mathbf{k}), \qquad \text{with} \qquad \mathbf{L}(\mathbf{k}) = -i\mathbf{k} \times \nabla_k \, .$$

- Prove that $\mathbf{J} |\psi\rangle = \int d\Omega_1 d\Omega_2 \, \{[\mathbf{L}(\mathbf{k}_1) + \mathbf{L}(\mathbf{k}_2)] f(\mathbf{k}_1, \mathbf{k}_2)\} \, a^\dagger(\mathbf{k}_1) a^\dagger(\mathbf{k}_2) |0\rangle$.
- How does this simplify in the center of mass frame where

$$f(\mathbf{k}_1, \mathbf{k}_2) = \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) g(\mathbf{k}_1) \, ?$$

- Take $\mathbf{k_1} = |\mathbf{k_1}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\phi)$,and choose the wavefunction to be

$$g(\mathbf{k}_1) = (2\pi)\delta(|\mathbf{k_1}| - k)\sin(\theta) e^{-i\phi}$$

  where k is a positive real number. Compute the action of J^3 on $|\psi\rangle$.

- What is the total angular momentum of this state?
  Hint: It might help to use the ladder operators $J^\pm = J^1 \pm iJ^2$.
- How would the action of the angular momentum operator $\mathbf{J}$ on an arbitrary n-particle state look like?
- Bonus question: Can you comment on what changes if we replace the scalar field by a Dirac field?