RELATIVITY AND COSMOLOGY II

Solutions to Problem Set 9 18th April 2025

1. Freeze-out of $p^+e^- \to n\nu_e$

1. In equilibrium, right before the decoupling of $p^+e^- \to n\nu_e$, we have
$$\mu_n = \mu_p + \mu_e - \mu_\nu. \tag{1}$$
As shown in exercise 3 of Problem Set 7, the difference in concentrations of electrons and positrons scales as
$$n_{e^-} - n_{e^+} \sim \mu_e T^2, \tag{2}$$
hence
$$\frac{\mu_e}{T} \sim \frac{n_{e^-} - n_{e^+}}{T^3} = \frac{n_p}{T^3} \sim \eta \sim 10^{-10}, \tag{3}$$
where in the second equality the electric neutrality of the plasma was used. We see that
$$\frac{\mu_e}{T} \sim 10^{-9}. \tag{4}$$
Since we assume that the lepton asymmetry is on the order of the baryon asymmetry,
$$\frac{n_\nu - n_{\bar\nu}}{T^3} \sim \eta \sim 10^{-9}, \tag{5}$$
we have by the same reasoning
$$\frac{\mu_\nu}{T} \sim \eta. \tag{6}$$
From Eq. (1) it now follows that
$$\mu_n \approx \mu_p. \tag{7}$$

2. At the moment of nucleosynthesis
$$\left.\frac{n_n}{n_p}\right|_{NS} \approx \left.\frac{n_n}{n_p}\right|_{*} e^{-\frac{t_{NS}}{\tau_n}} \approx e^{-\frac{Q}{T^*} - \frac{\mu_{\nu_e}}{T^*} - \frac{t_{NS}}{\tau_n}}, \tag{8}$$
where $\left.\frac{n_n}{n_p}\right|_* = e^{-\frac{Q}{T^*} - \frac{\mu_{\nu_e}}{T^*}}$ is the ratio at the moment of freeze-out of $p^+e^- \to n\nu_e$, at $T^* \approx 0.8\,\text{MeV}$. (This ratio approximately equals to 1/5 if the neutrinos' chemical potential is negligible.) Here, we used the notations for $Q = m_n - m_p \approx 1.3\,\text{MeV}$, the time of nucleosynthesis $t_{NS} \approx 4.5\,\text{min}$, the neutron lifetime $\tau_n \approx 15\,\text{min}$, and we neglected the chemical potential μ_e of the electron. The ratio $\frac{n_n}{n_p}$ determines the abundance of ^{4}He:
$$X_{^4\text{He}} = \frac{2\left.\frac{n_n}{n_p}\right|_{NS}}{1 + \left.\frac{n_n}{n_p}\right|_{NS}}. \tag{9}$$

The difference between the abundances for zero and non-zero μ_{ν_e} is

$$\begin{aligned}\delta X &\equiv X_{^4\mathrm{He}} - X_{^4\mathrm{He}}\Big|_{\mu_{\nu_e}=0} \\ &\approx \frac{2\frac{n_n}{n_p}\left(1-\frac{\mu_{\nu_e}}{T^*}\right)}{1+\frac{n_n}{n_p}\left(1-\frac{\mu_{\nu_e}}{T^*}\right)} - \frac{2\frac{n_n}{n_p}}{1+\frac{n_n}{n_p}} \\ &\approx -\frac{\mu_{\nu_e}}{T^*}\frac{2n_n/n_p}{(1+n_n/n_p)^2}, \end{aligned} \tag{10}$$

assuming the ratio μ_{ν_e}/T^* is small. Here n_n/n_p is the ratio at the moment of nucleosynthesis for zero chemical potential of neutrinos, i.e., $\frac{n_n}{n_p} \approx \frac{1}{7}$.

So that gives

$$\delta X = -\frac{7}{32}\frac{\mu_{\nu_e}}{T^*}. \tag{11}$$

Given that $|\delta X| \lesssim 0.05 X_{^4\mathrm{He}} \approx 1/80$, we have for $T^* \approx 0.8\,\mathrm{MeV}$

$$\mu_{\nu_e} \lesssim \frac{2}{35}T^* \approx 46\,\mathrm{keV}. \tag{12}$$

2. Variation of ^{4}He abundance

We know that the abundance of ^{4}He is

$$X_{^4\mathrm{He}} = \frac{2y}{1+y}, \qquad y \equiv \frac{n_n}{n_p}\Big|_{NS} \approx \exp\left(-\frac{Q}{T^*} - \frac{t_{NS}}{\tau_n}\right), \tag{13}$$

where $Q = m_n - m_p$, T^* is the neutron freeze-out temperature, t_{NS} is the time of nucleosynthesis, and τ_n is the free neutron lifetime. This implies:

$$\frac{\delta X_{^4\mathrm{He}}}{X_{^4\mathrm{He}}} = \frac{1}{1+y}\frac{\delta y}{y} \approx \frac{7}{8}\frac{\delta y}{y}. \tag{14}$$

1. First we want to see how the abundance $X_{^4\mathrm{He}}$ is affected when the number of the relativistic degrees of freedom changes. To begin with, we would like to mention that the number of relativistic degrees of freedom enters two quantities: the decoupling temperature T^* and the time of the nucleosynthesis t_{NS}. Thus we get

$$\frac{\delta y}{y} = \frac{Q}{T^{*2}}\delta T^* - \frac{1}{\tau_n}\delta t_{NS}. \tag{15}$$

To find the change in the temperature, we use the fact that the reactions under consideration decouple at a temperature very close to the neutrino decoupling temperature, so

$$G_F^2 T^{*5} \sim \frac{T^{*2}\sqrt{g^*}}{\tilde{M}}, \quad \Rightarrow \quad T^* \propto (g^*)^{1/6}. \tag{16}$$

Therefore,

$$\frac{\delta T^*}{T^*} = \frac{1}{6}\frac{\delta g^*}{g^*}. \tag{17}$$

To find the change in the time of nucleosynthesis, we should note that the *temperature* of nucleosynthesis $T_{NS} \approx 70\,\mathrm{keV}$ does not depend on g^* and is determined by the deuterium binding energy. Then, the corresponding time can be expressed as follows:

$$t_{NS} = \frac{1}{2H_{NS}} = \frac{\tilde{M}}{T_{NS}^2\sqrt{g^*}}, \quad \Rightarrow \quad t_{NS} \propto (g^*)^{-1/2} . \tag{18}$$

Thus,

$$\frac{\delta t_{NS}}{t_{NS}} = -\frac{1}{2}\frac{\delta g^*}{g^*} . \tag{19}$$

Plugging Eqs. (17) and (19) into Eq. (15) and then using Eq. (14), we get

$$\frac{\delta X_{^4\mathrm{He}}}{X_{^4\mathrm{He}}} = \frac{7}{8}\left(\frac{1}{6}\frac{Q}{T^*} + \frac{1}{2}\frac{t_{NS}}{\tau_n}\right)\frac{\delta g^*}{g^*} \simeq \frac{7}{8}\left(\frac{1}{6}\frac{1.3}{0.8} + \frac{1}{2}\frac{4.5}{15}\right)\frac{\delta g^*}{g^*} \simeq 0.37\frac{\delta g^*}{g^*} . \tag{20}$$

If we consider an additional massless neutrino (and its antiparticle), we find that

$$\frac{\delta X_{^4\mathrm{He}}}{X_{^4\mathrm{He}}} \simeq 0.37\frac{\frac{7}{8}(2)}{2+\frac{7}{8}(4+6)} \simeq 0.06 = 6\%. \tag{21}$$

Thus, we get an upper limit on the number of massless neutrinos:

$$\#\nu < 4 .$$

A more detailed calculation combined with the uncertainty for the experimental measurements of $X_{^4\mathrm{He}}$ confirms this result.

2. In complete analogy with the above, we can calculate how $X_{^4\mathrm{He}}$ varies when we change Q

$$\frac{\delta X_{^4\mathrm{He}}}{X_{^4\mathrm{He}}} = \frac{7}{8}\frac{\delta y}{y} = -\frac{7}{8}\frac{Q}{T^*}\frac{\delta Q}{Q} \simeq -1.4\frac{\delta Q}{Q} \simeq -0.14 = -14\% . \tag{22}$$

3. Again, in the same way as in points 1 and 2, we can compute how $X_{^4\mathrm{He}}$ is affected by the change of neutron's lifetime τ_n:

$$\frac{\delta X_{^4\mathrm{He}}}{X_{^4\mathrm{He}}} = \frac{7}{8}\frac{\delta y}{y} = \frac{7}{8}\frac{t_{NS}}{\tau_n}\frac{\delta\tau_n}{\tau_n} \simeq 0.26\frac{\delta\tau_n}{\tau_n} \simeq -0.026 = -2.6\% , \tag{23}$$

where we used $\frac{\delta\tau_n}{\tau_n} = -0.1$, $t_{\mathrm{NS}} \approx 270\,\mathrm{s}$ and $\tau_n = 881\,\mathrm{s}$ (the last two numbers come from the lecture).

3.* Post-freezing antiprotons density

As we know from the lectures, the number density obeys the Boltzmann equation. Now we are going to see what happens to the number density $\bar{n}$ of antiprotons. For simplicity we will take into account only protons, antiprotons and photons as the temperature we will find will be below pion mass. The number of antiprotons in a comoving volume changes due to annihilations with protons and due to the inverse process of proton-antiproton pair creation. The annihilation contributes

$$\frac{d\bar{n}}{dt} + 3H\bar{n} = -\Gamma_{ann}\ \bar{n}\ , \tag{24}$$

where the rate of annihilation Γ_{ann} is

$$\Gamma_{ann} = \sigma_{ann}\ v\ n\ , \tag{25}$$

with n the number density of the target particles (in our case the protons), v the velocity and σ_{ann} the annihilation cross section. It can be shown that for non-relativistic particles $\sigma_{ann} = \sigma_0/v$. where σ_0 is proportional to the square of the Compton wavelength of the proton[1]

$$\sigma_0 \sim \lambda_p^2 \sim 25\ \text{GeV}^{-2}\ . \tag{26}$$

The pair production contributes

$$\frac{d\bar{n}}{dt} + 3H\bar{n} = +\Gamma_{prod}\ \bar{n}^{eq}\ , \tag{27}$$

where the rate of these interactions is

$$\Gamma_{prod} = \sigma_0\ n^{eq}\ . \tag{28}$$

From the above equations the Boltzmann equation for antiproton is

$$\frac{d\bar{n}}{dt} + 3H\bar{n} = -\sigma_0\left[n\ \bar{n} - n^{eq}\ \bar{n}^{eq}\right]\ , \tag{29}$$

where the first and second terms correspond to the annihilation and production processes, respectively. The equation for proton can be written in the same manner.
The baryon number is conserved below $T \approx 100$ GeV (see next lectures).

$$\frac{n-\bar{n}}{n_\gamma} = \eta = \text{const} \quad \rightarrow \quad n = \bar{n} + \eta\, n_\gamma, \tag{30}$$

where $\eta = 6\times 10^{-10}$. Substituting Eq. (30) to Eq. (29), the Boltzmann equation is

$$\frac{d\bar{n}}{dt} + 3H\bar{n} = -\sigma_0\,\eta\, n_\gamma \bar{n} + \sigma_0\, n^{eq}\,\bar{n}^{eq} - \sigma_0\,\bar{n}^2. \tag{31}$$

Assuming that when all reactions freeze the particles are non-relativistic ($T \ll m_p$), their (equilibrated) number densities are

$$\begin{aligned} n^{eq} &= 2\left(\frac{m_p T}{2\pi}\right)^{3/2} \exp\left(\frac{-m_p+\mu_p}{T}\right)\ , \\ \bar{n}^{eq} &= 2\left(\frac{m_p T}{2\pi}\right)^{3/2} \exp\left(\frac{-m_p-\mu_p}{T}\right)\ . \end{aligned} \tag{32}$$

As the annihilation proceeds $\bar{n} \lesssim \eta\, n_\gamma$ below a temperature T^*[2]. For such a temperature region

$$\bar{n}^{eq}\, n^{eq} = 4\left(\frac{m_p T}{2\pi}\right)^3 e^{-\frac{2m_p}{T}} \quad \rightarrow \quad \bar{n}^{eq} \approx \frac{4}{\eta\, n_\gamma}\left(\frac{m_p T}{2\pi}\right)^3 e^{-\frac{2m_p}{T}}, \tag{33}$$

[1]We remind that $(\Gamma_{ann})^{-1}$ is the time needed for an antiproton with cross section σ_{ann} to meet one proton.
[2]$T^* = 36$ MeV is estimated numerically from $\bar{n}^{eq} = \eta\, n_\gamma$.

and the last term in Eq. (31) can be neglected when $\bar{n} \ll \eta\, n_\gamma$. After removing this term, Eq. (31) can be written as

$$\frac{d\bar{n}}{dt} + 3H\bar{n} \approx -\sigma_0\, \eta\, n_\gamma (\bar{n} - \bar{n}^{eq}), \tag{34}$$

or

$$\frac{d\bar{\eta}}{dt} \approx -\sigma_0\, \eta\, n_\gamma (\bar{\eta} - \bar{\eta}^{eq}), \tag{35}$$

where $\bar{\eta} \equiv \bar{n}/n_\gamma$.
Once Eq. (35) is rewritten as

$$\bar{\eta} = \bar{\eta}^{eq} - \frac{1}{\sigma_0\, \eta\, n_\gamma} \frac{d\bar{\eta}}{dt}, \tag{36}$$

the iterative solution is

$$\bar{\eta} = \bar{\eta}^{eq} - \frac{1}{\sigma_0\, \eta\, n_\gamma} \frac{d\bar{\eta}^{eq}}{dt} + \cdots . \tag{37}$$

The derivative of $\bar{\eta}$ in the second term is

$$\frac{d\bar{\eta}^{eq}}{dt} = \frac{d\bar{\eta}^{eq}}{dT} \frac{dT}{dt} \approx \bar{\eta}^{eq} \left(-\frac{2m_p}{T^2} \right) HT, \tag{38}$$

where we used $2t = M_0/T^2 = H^{-1} \to dT/dt = -HT$, and the condition for the second term to be smaller than the first term is

$$\frac{2m_p}{T_1} H \lesssim \sigma_0\, \eta\, n_\gamma \quad \to \quad T_1 \gtrsim \left(\frac{m_p}{\sigma_0\, \eta\, M_0} \frac{\pi^2}{\zeta(3)} \right)^{\frac{1}{2}} \approx 27 \text{ keV}, \tag{39}$$

where $n_\gamma = (2\zeta(3)/\pi^2) T^3$. It means for $T \lesssim T_1$ the abundance of antiproton deviates from that in equilibrium. From (33) at this temperature T_1

$$\bar{\eta}|_{T_1} \approx 7 \times 10^{-3\times 10^4}. \tag{40}$$

After this moment we can neglect the second term in Eq. (35), and have

$$\frac{d\bar{\eta}}{dt} \approx -\sigma_0\, \eta\, n_\gamma \bar{\eta}. \tag{41}$$

It has a solution as

$$\bar{\eta} \approx \bar{\eta}|_{T_1} \exp \left[-\sigma_0\, \eta\, \frac{2\zeta(3)}{\pi^2} M_0 (T_1 - T) \right], \tag{42}$$

and the freeze-out temperature of antiproton is given by

$$H(T_2) = \sigma_0\, \eta\, n_\gamma(T_2) \quad \to \quad T_2 \approx \left(\sigma_0\, \eta\, \frac{2\zeta(3)}{\pi^2} M_0 \right)^{-1} \approx 4 \times 10^{-10} \text{ GeV} \approx 0.4 \text{ eV}. \tag{43}$$

For $T < T_2$ we have

$$\bar{\eta}|_{T<T_2} \approx \bar{\eta}|_{T_1} \exp \left[-\sigma_0\, \eta\, \frac{2\zeta(3)}{\pi^2} M_0 T_1 \right] \approx 4 \times 10^{-6\times 10^4}. \tag{44}$$

This means that the Universe practically contains no antiprotons because $n/n_\gamma \simeq \eta \sim 10^{-10}$.
We show the numerical solution of (31) in Fig. 1, where we can see the above estimation is consistent with that.

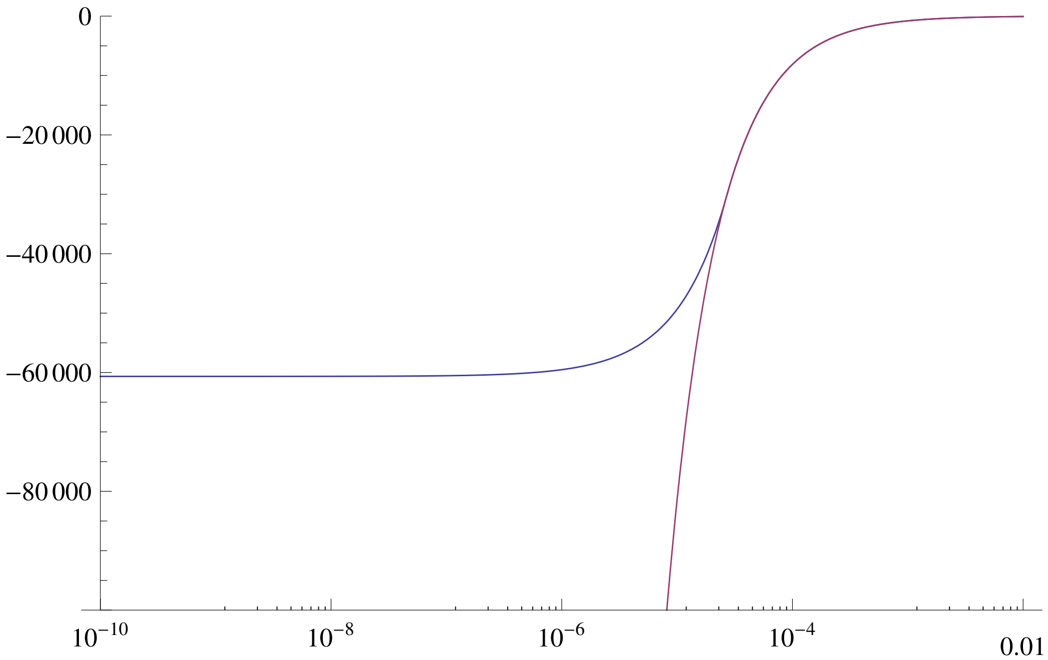

Figure 1: The horizontal and vertical axes are T [GeV] and $\log_{10}(\bar{n}/n_\gamma)$, respectively. Blue line is the numerical solution and red line corresponds to equilibrium number density.