RELATIVITY AND COSMOLOGY II

Solutions to Problem Set 8 19th April 2024

1. Fraction of free protons

The Saha equation in thermal equilibrium reads as

$$n_H = n_e n_p \left(\frac{m_e T}{2\pi}\right)^{-3/2} e^{I/T} \ , \tag{1}$$

where $I = m_p + m_e - m_H = 13.6\,\text{eV}$ is the binding energy of the hydrogen atom. For an electrically neutral plasma, we have $n_e = n_p$. Introducing the fraction of free protons $x \equiv \frac{n_p}{n_B}$, where $n_B = n_p + n_H$ is the number of baryons, we can write the above as

$$\frac{1-x}{x^2} = K, \qquad K = n_B \left(\frac{m_e T}{2\pi}\right)^{-3/2} e^{I/T} = \frac{2\zeta(3)}{\sqrt{\pi}}\eta\left(\frac{2T}{m_e}\right)^{3/2} e^{I/T} \ , \tag{2}$$

where we used the relation $n_B = \eta\, n_\gamma$ with the baryon-to-photon ratio

$$\eta \equiv \frac{n_B}{n_\gamma} \approx 10^{-10} \ , \tag{3}$$

and the photon number density

$$n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3 \ . \tag{4}$$

Therefore, the fraction of free protons satisfies the quadratic equation

$$K x^2 + x - 1 = 0, \tag{5}$$

whose solutions are

$$x_{1,2} = \frac{-1 \pm \sqrt{1+4K}}{2K}. \tag{6}$$

Since x must be in a range [0, 1], the only physical solution is that with a "plus" sign, i.e.,

$$x = \frac{-1 + \sqrt{1+4K}}{2K}. \tag{7}$$

Thus, for any given temperature T one gets a unique solution for x. In particular, for $T = T_d = 0.25\,\text{eV}$, we have $K \approx 55\,440$ and

$$x = \frac{-1 + \sqrt{1+4K}}{2K} \approx \frac{1}{\sqrt{K}} \approx 4.2 \times 10^{-3}. \tag{8}$$

Indeed, the fraction of free protons at that time is negligibly small.

2. Decoupling and concentration

1. Just like in the lecture, the whole idea here is to use the conservation of entropy to find out how the temperature changes. The difference is that now we have an extra species ϕ, which decouples at $T_d \approx 150\,\mathrm{MeV}$. Note that since the muon decoupling temperature $T_{d,\mu} \approx \frac{m_\mu}{40} \approx 2.5\,\mathrm{MeV}$, muons are still around in the plasma when ϕ decouples[1].

   Therefore, we arrive at the following picture. In the beginning, we have a plasma that contains the new species ϕ (with 1 d.o.f.), photons (2 d.o.f.), two charged leptons (electron and muon, 4 d.o.f. each), and three neutrinos (2 d.o.f. each). So then, when everything is still in equilibrium (forming a plasma), we have for the entropy density

$$s = \frac{2\pi^2}{45} \underbrace{\left(2 + 1 + \frac{7}{8}\,(2 \cdot 4 + 3 \cdot 2)\right)}_{61/4} T^3, \tag{9}$$

   where T stands for the temperature of the whole plasma.

   When the temperature has dropped to $T = T_d = 150\,\mathrm{MeV}$, the particle ϕ decouples from the plasma. From then on we have

$$s = s_1 + s_2, \quad s_1 = \frac{2\pi^2}{45} \underbrace{\left(2 + \frac{7}{8}\,(2 \cdot 4 + 3 \cdot 2)\right)}_{57/4} T^3, \quad s_2 = \frac{2\pi^2}{45} T_\phi^3, \tag{10}$$

   and these two terms evolve independently, i.e., the entropies $S_1 = s_1 V$ and $S_2 = s_2 V$ are conserved separately. Here T denotes the temperature of the plasma, and T_ϕ the temperature of particle ϕ. As long as nothing dramatic is happening, they are still equal: they still fall off with the expansion of the universe. However, at one point (around $T_{d,\mu} \simeq 2.5\,\mathrm{MeV}$), the μ^+ and μ^- get annihilated (i.e., the reaction $\mu^+\mu^- \leftrightarrow 2\gamma$ goes out of equilibrium, because the temperature is too low now for the production of a muon pair), and we are left with

$$s_1 = \frac{2\pi^2}{45} \underbrace{\left(2 + \frac{7}{8}\,(1 \cdot 4 + 3 \cdot 2)\right)}_{43/4} T^3, \qquad s_2 = \frac{2\pi^2}{45} T_\phi^3. \tag{11}$$

   At this moment, the effective number of degrees of freedom in subsystem "1" has decreased by a factor of 43/57, which means that T^3 has to increase by a factor of 57/43, just to keep the entropy $S_1 = s_1 V$ constant. So this is the moment when the temperature T of the plasma begins to differ from the temperature T_ϕ of the species ϕ.

   Next comes the decoupling of the neutrinos, at temperature $T_{d,\nu} \approx 2\,\mathrm{MeV}$

$$s_1 = s_{11} + s_{12}, \quad s_{11} = \frac{2\pi^2}{45} \underbrace{\left(2 + \frac{7}{8}\,(1 \cdot 4)\right)}_{11/2} T^3, \quad s_{12} = \frac{2\pi^2}{45} \cdot \underbrace{\frac{7}{8}\,(3 \cdot 2)}_{21/4} T_\nu^3. \tag{12}$$

[1] For derivation of muon decoupling temperature, follow the notes of Lecture 7 – there is analogous derivation for electrons.

Starting from this point, the electron-positron-photon subsystem evolves independently of the neutrino subsystem. The corresponding entropies $S_{11} = s_{11}V$ and $S_{12} = s_{12}V$ are conserved separately. Finally, at $T_{d,e} \simeq 10\,\text{keV}$, electrons and positrons annihilate each other:

$$s_{11} = \frac{2\pi^2}{45} \cdot 2 \cdot T_\gamma^3, \quad s_{12} = \frac{2\pi^2}{45} \cdot \frac{7}{8}\,(3 \cdot 2)\,T_\nu^3, \quad s_2 = \frac{2\pi^2}{45} T_\phi^3. \tag{13}$$

From now on we only have photons left in the plasma, that's why its temperature is denoted as T_γ. Once again: at the moment of decoupling of electrons the effective number of degrees of freedom in subsystem "11" goes down by a factor of 4/11, so T^3 increases by a factor of 11/4.

Already from this schematic analysis, we may conclude that at muon-antimuon annihilation the temperature of all particles which were in equilibrium with muons increased in comparison to the temperature of species ϕ (already decoupled) by a factor of $(57/43)^{1/3}$. Later, at electron-positron annihilation, photons (as no other particles were in equilibrium with electrons and positrons) were "heated up" again by a factor of $(11/4)^{1/3}$. Thus, in the end, the temperatures are related as follows:

$$T_\phi = \left(\frac{43}{57} \cdot \frac{4}{11}\right)^{1/3} T_\gamma. \tag{14}$$

However, let us derive this relation in a more rigorous way. Let us choose three moments in the thermal history: (1) temperature T_{in} (the same for all particles) which is below the decoupling temperature $T_d = 150\,\text{MeV}$ and above the muon-antimuon annihilation freeze-out temperature 2.5 MeV; (2) temperature T_{mid} (different for species ϕ and all other particles) somewhere below the neutrino decoupling $T_{d,\nu} = 2\,\text{MeV}$ and above the electron-positron annihilation freeze-out temperature 10 keV; (3) temperature T_{out} (different for ϕ, ν, and γ) below 10 keV. We know that between (1) and (3) entropies S_1 and S_2 are conserved separately while between (2) and (3) entropies S_{11} and S_{12} are conserved separately. Taking into account that $S = sV$, $V \propto a^3$, we have the following equations:

$$S_{1,\text{in}} = S_{1,\text{out}} \quad \Leftrightarrow \quad \frac{57}{4} T_{\text{in}}^3 a_{\text{in}}^3 = 2T_{\gamma,\text{out}}^3 a_{\text{out}}^3 + \frac{7}{8} \cdot 6\, T_{\nu,\text{out}}^3 a_{\text{out}}^3, \tag{15}$$

$$S_{2,\text{in}} = S_{2,\text{out}} \quad \Leftrightarrow \quad T_{\text{in}}^3 a_{\text{in}}^3 = T_{\phi,\text{out}}^3 a_{\text{out}}^3, \tag{16}$$

$$S_{11,\text{mid}} = S_{11,\text{out}} \quad \Leftrightarrow \quad \frac{11}{2} T_{\text{mid}}^3 a_{\text{mid}}^3 = 2T_{\gamma,\text{out}}^3 a_{\text{out}}^3, \tag{17}$$

$$S_{12,\text{mid}} = S_{12,\text{out}} \quad \Leftrightarrow \quad \frac{21}{4} T_{\text{mid}}^3 a_{\text{mid}}^3 = \frac{21}{4} T_{\nu,\text{out}}^3 a_{\text{out}}^3. \tag{18}$$

Taking the ratio of Eqs. (17) and (18), we get

$$T_{\nu,\text{out}} = \left(\frac{4}{11}\right)^{1/3} T_{\gamma,\text{out}}, \tag{19}$$

which you derived during the lecture. Further, taking the ratio of Eqs. (16) and (15), we obtain

$$T_{\phi,\text{out}}^3 = \frac{4}{57}\left(2T_{\gamma,\text{out}}^3 + \frac{21}{4} T_{\nu,\text{out}}^3\right) = \frac{4}{57}\left(2 + \frac{21}{4}\frac{4}{11}\right) T_{\gamma,\text{out}}^3 = \frac{4}{57}\frac{43}{11} T_{\gamma,\text{out}}^3, \tag{20}$$

which is equivalent to Eq. (14) above.

2. Next, we want to find the number density of the particle ϕ today. For this purpose, we use the fact that below 10 keV the temperatures of photons, T_γ, and ϕ particles, T_ϕ, both fall with the scale factor as $\sim 1/a$, therefore, their ratio remains constant. Then, at present time

$$T_{\phi,0} = \left(\frac{43}{57} \cdot \frac{4}{11}\right)^{1/3} T_{\gamma,0} \approx 0.65\, T_{\gamma,0} \approx 1.77\,\mathrm{K} \approx 1.52 \times 10^{-14}\,\mathrm{GeV}. \tag{21}$$

Since particles ϕ decoupled when they were ultrarelativistic, their distribution function keeps the ultrarelativistic Bose-Einstein form until the present time even though the temperature is well below the particle's mass. Therefore, the number density is given by the same UR formula

$$n_{\phi,0} = \frac{\zeta(3)}{\pi^2} T_{\phi,0}^3 = 4.3 \times 10^{-40}\,\mathrm{GeV}^3 \approx 56\,\mathrm{cm}^{-3}. \tag{22}$$

3. For the associated energy density we then have ($m_\phi \approx 100\,\mathrm{eV}$)

$$\rho_\phi = m_\phi \cdot n_\phi = 5.6 \times 10^9\,\mathrm{eV/m}^3 = 9.9 \times 10^{-27}\,\mathrm{kg/m}^3. \tag{23}$$

Meanwhile, the critical density is given by

$$\rho_{\mathrm{cr}} = \frac{3H_0^2}{8\pi G} \approx 8.5 \times 10^{-27}\,\mathrm{kg/m}^3. \tag{24}$$

Thus, the cosmological abundance of the particles ϕ would be

$$\Omega_\phi = \frac{\rho_\phi}{\rho_{\mathrm{cr}}} \approx 1.16 > 1. \tag{25}$$

Therefore, if the particle ϕ would exist, it would be overabundant in the Universe. This is inconsistent with observations.

3. Lee-Weinberg bound

1. When the reaction rate of the neutrino Γ is comparable to Hubble parameter H the neutrino stops interacting

$$\Gamma(T^*) \sim H(T^*) \ , \tag{26}$$

where T^* is the temperature of decoupling. The rate of the reaction can be approximated by $\Gamma \sim n_{\nu_4}\langle\sigma v\rangle$. The number density that appears in the relationship $\Gamma \sim n\langle\sigma v\rangle$ is the density of target particles. It is assumed that the most important reaction to keep the new neutrinos ν_4 at thermal equilibrium is the annihilation of neutrino with its own antiparticle. We would then put $n_{\bar\nu_4}$. But it is also assumed that there is no asymmetry between neutrinos and antineutrinos, so $n_{\bar\nu_4} = n_{\nu_4}$.

The Hubble parameter during the domination of radiation can be expressed as $H = 1.66\sqrt{g^*(T)}\frac{T^2}{M_{\rm pl}}$, so the relationship (26) becomes

$$n_{\nu_4}(T^*)\langle\sigma v\rangle = \sqrt{g^*(T^*)}\frac{T^{*2}}{\tilde M} \ , \tag{27}$$

where we defined the constant $\tilde M = M_{\rm pl}/1.66$.

2. The constraint to be imposed is that the present additional energy density of the neutrino ν_4 must not exceed the density of dark matter

$$\rho_{\nu_4} = 2m_{\nu_4}n_{\nu_4}(T_0) \leq \Omega_{DM}\rho^0_{\rm cr} \ . \tag{28}$$

The equality is satisfied if all the dark matter is given by neutrinos ν_4.

Once the neutrinos stop interacting, the number of neutrinos per comoving volume remains constant $n_{\nu_4}a^3 = $ const. In addition, we know that the total entropy per comoving volume is conserved, i.e. $sa^3 = $ const. By combining the two conservation laws we can express the density of neutrinos today as

$$n_{\nu_4}(T_0) = n_{\nu_4}(T^*)\frac{s(T_0)}{s(T^*)} = n_{\nu_4}(T^*)\frac{g^*(T_0)}{g^*(T^*)}\left(\frac{T_0}{T^*}\right)^3 \ . \tag{29}$$

Using this relationship we can write (28) like

$$2m_{\nu_4}n_{\nu_4}(T^*)\frac{g^*(T_0)}{g^*(T^*)}\left(\frac{T_0}{T^*}\right)^3 \leq \Omega_{DM}\rho^0_{\rm cr} \ . \tag{30}$$

The combination of relations (27) and (30) allows us to determine the temperature of decoupling and the limits on the mass of ν_4. We will be interested in two limiting cases. The neutrino is light and decouples while still relativistic, or it is very heavy and it is already non-relativistic at decoupling.

3. Light neutrino ν_4: If the neutrino ν_4 is relativistic at the time of decoupling, the density at T^* is given by

$$n_{\nu_4}(T^*) = \frac{3\zeta(3)}{4\pi^2}T^{*3} \ .$$

The thermal average of the product of the cross section and the relative velocity is

$$\langle\sigma v\rangle \sim G_F^2T^{*2} \ .$$

With these two relations, (27) gives us the decoupling temperature

$$T^* = \left(\frac{4\pi^2 \sqrt{g^*(T^*)}}{3\zeta(3) G_F^2 \tilde{M}} \right)^{\frac{1}{3}} \simeq 2.2\ (g^*(T^*))^{1/6}\ \text{MeV}\ . \tag{31}$$

We know that g^* is 106.75 when all species of Standard Model are relativistic and about 3 when there is only photons and massless neutrinos. For these values, $(g^*(T^*))^{1/6}$ is of the order of 1. Our estimate of T^* is then

$$T^* \sim 2\,\text{MeV}\ . \tag{32}$$

At this temperature, the relativistic species are γ, ν_e, $\bar{\nu}_e$, $\bar{\nu}_\mu$, ν_μ, $\bar{\nu}_\tau$, ν_τ, $\bar{\nu}_4$, ν_4, e^+, e^-. Note that neutrinos (and antineutrons) carry one degree of freedom, since they come only with lefthanded polarization, while electrons (and positrons) carry two degrees of freedom, since they come with both left- and righthanded polarizations. We then get

$$g^*(T^*) = 2 + \frac{7}{8}(1 \cdot 4 + 3 \cdot 2 + 2) = \frac{25}{2}\ . \tag{33}$$

Then, the estimate for the decoupling temperature can be made more accurately:

$$T^* \approx 2.2 \cdot (25/2)^{1/6} \approx 3.4\,\text{MeV}. \tag{34}$$

At the present time, the only particles that are still relativistic are photons and massless (standard) neutrinos. Neutrinos ν_4 should also be included because they decoupled being ultrarelativistic and since then their entropy is conserved (note that the temperature of all neutrinos differs from that of photons, see problem 2 and lecture notes). Thus

$$g^*(T_0) = 2 + \frac{7}{8} \cdot 8 \cdot \left(\frac{T_\nu}{T_0} \right)^3 = 2 + \frac{7}{8} \cdot 8 \cdot \frac{4}{11} = \frac{50}{11}\ . \tag{35}$$

We substitute (33) and (35) in (30) and solve for m_{ν_4} to find

$$m_{\nu_4} \leq \frac{2\pi^2}{3\zeta(3)} \frac{g^*(T^*)}{g^*(T_0)} \frac{\Omega_{DM} \rho_{\text{cr}}^0}{T_0^3} \simeq 10.7\,\text{eV}. \tag{36}$$

4. Heavy neutrino ν_4: If the new neutrino is non-relativistic when decoupling, the density is given by

$$n_{\nu_4}(T^*) = \left(\frac{m_{\nu_4} T^*}{2\pi}\right)^{3/2} \exp\left(-\frac{m_{\nu_4}}{T^*}\right) . \tag{37}$$

We can already see that in this case a larger mass corresponds to a smaller density. We therefore expect to find a lower limit for the new neutrino mass. The thermal average product of the cross section and the relative velocity is now

$$\langle \sigma v \rangle \sim G_F^2 m_{\nu_4}^2 .$$

This result is not entirely trivial. One might expect the presence of a factor $(T/m_{\nu_4})^n$. A detailed calculation shows that for the case of a Dirac neutrino, the dominant contribution is independent of T. The relations (27) and (30) then become

$$m_{\nu_4}^{7/2} T^{*-1/2} \exp(-m_{\nu_4}/T^*) = \frac{(2\pi)^{3/2}\sqrt{g^*(T^*)}}{G_F^2 \tilde{M}} \equiv a\sqrt{g^*(T^*)}, \tag{38}$$

$$m_{\nu_4}^{5/2} T^{*-3/2} \exp(-m_{\nu_4}/T^*) = \frac{(2\pi)^{3/2}\Omega_{DM}\rho_{\rm cr}^0}{2T_0^3} \frac{g^*(T^*)}{g^*(T_0)} \equiv b\, g^*(T^*) , \tag{39}$$

$$a = 1.6 \times 10^{-8}\,\mathrm{GeV}^3, \qquad b = 1.4 \times 10^{-9}\,\mathrm{GeV}, \tag{40}$$

where the inequality was replaced by an equality to simplify the calculations. Here we used $g^*(T_0) = 2 + (7/8) \cdot 6 \cdot (4/11) = 43/11$ because only 3 neutrino species contribute to the entropy, the forth one decoupled already being non-relativistic.

Unfortunately we cannot find an analytic solution to the above equations. We start by making the change of variables $x = \frac{1}{2}\frac{m_{\nu_4}}{T^*}$. The equations become

$$(2x)^{1/2} m_{\nu_4}^3 \exp(-2x) = a\sqrt{g^*(T^*)} , \tag{41}$$

$$(2x)^{3/2} m_{\nu_4} \exp(-2x) = b\, g^*(T^*) . \tag{42}$$

We can eliminate m_{ν_4} to find an equation for x

$$\frac{e^x}{x} = \left(\frac{16a}{b^3}\right)^{1/4} [g^*(T^*)]^{-5/8} \approx 96\,600 [g^*(T^*)]^{-5/8} \equiv K . \tag{43}$$

By taking the logarithm, we obtain

$$x - \ln x = \ln K . \tag{44}$$

By assumption we have $x \gg 1$. We will try to solve the equation by iterations. As $\ln x \ll x$, we can neglect the logarithm to find a first approximation

$$x_0 = \ln K . \tag{45}$$

To find the solution, we reinsert x_0 in the equation

$$x \simeq x_1 = \ln K + \ln x_0 = \ln(K \ln K) . \tag{46}$$

By inserting x_1 in Eq. (42) we obtain

$$T^* \simeq \frac{b g^*(T^*) e^{2x_1}}{(2x_1)^{5/2}} = \sqrt{\frac{a}{2bx_1}} [g^*(T^*)]^{-1/4} . \tag{47}$$

As before, we must analyze the dependence of T^* on $g^*(T^*)$. In this case the result is more sensitive to the value of g^*. For a first estimate let us take $g^*(T^*) = 20$. Then, $K \approx 14\,900$ and $x_1 \approx 12$. The decoupling temperature then reads $T^* \simeq 320\,\text{MeV}$. This is just above the QCD phase transition. Therefore, we can take

$$g^*(T^*) = 2_\gamma + 8 \cdot 2_g + \frac{7}{8}(2 \cdot 4_l + 3 \cdot 2_\nu + 3 \cdot 12_q) = 61.75\,. \tag{48}$$

With this new value we can update $K = 7340$, $x_1 \simeq 11.3$, and

$$T^* \approx 250\,\text{MeV}. \tag{49}$$

It is still above the QCD phase transition. Thus, this result is self-consistent. Then, we can then find the limit on the mass m_{ν_4}

$$m_{\nu_4} \geq 2x_1 T^* \simeq 5.7\,\text{GeV}\ . \tag{50}$$