RELATIVITY AND COSMOLOGY II

Solutions to Problem Set 7 4th April 2025

1. Blackbody radiation

Compared with some initial time t_0, the lengths, in particular the wavelength, get red-shifted

$$\lambda(t) = \lambda_0 \frac{R(t)}{R_0} ,$$

where $R(t)$ is the scale factor. Since the frequency is inversely proportional to the wavelength, we have

$$\omega(t) = \omega_0 \frac{R_0}{R(t)} .$$

Then,

$$\frac{dE^0_\omega}{d\omega_0} = \frac{V_0 \hbar}{\pi^2 c^3} \frac{\omega_0^3}{e^{\frac{\hbar\omega_0}{kT_0}} - 1} = \frac{V_0 \hbar}{\pi^2 c^3} \frac{\omega(t)^3 \left(\frac{R(t)}{R_0}\right)^3}{e^{\frac{\hbar\omega(t) R(t)}{kTR_0}} - 1} = \frac{V(t)\hbar}{\pi^2 c^3} \frac{\omega(t)^3}{e^{\frac{\hbar\omega(t)}{kT(t)}} - 1} = \frac{dE_\omega(t)}{d\omega(t)} , \quad (1)$$

since $V(t) = V_0 \frac{R^3(t)}{R_0^3}$.

Note that for the last two equalities, we had to impose $T(t) = T_0 \frac{R_0}{R(t)}$, which gives us the evolution of the temperature of radiation during the expansion of the universe. It is easy to see that the total entropy $S \propto VT^3$ is conserved

$$V_0 T_0^3 = V_0 T^3(t) \left(\frac{R(t)}{R_0}\right)^3 = V(t) T^3(t) .$$

2. Photon decoupling in numbers

1. The redshift is defined as

$$1 + z(t) = \frac{R_0}{R(t)} , \quad (2)$$

   where $R_0 = R(t_0)$ is the scale factor today. Since temperature redshifts as $T \sim R^{-1}$, we find that at the time of decoupling

$$z(t_d) = \frac{T_d}{T_0} - 1 \approx 1100 . \quad (3)$$

2. In order to compute the age of the Universe

$$t = \int dt ,$$

when photons decoupled, we have to express the above in terms of the redshift. We start from the definition of the Hubble parameter

$$H = \frac{\dot{R}}{R} = \frac{1}{R}\frac{dR}{dt} = \frac{1}{R}\frac{dR}{dz}\frac{dz}{dt} \ .$$

Using $1+z = \frac{R_0}{R}$, the above becomes

$$H = -\frac{1}{dt}\frac{dz}{(1+z)} \ .$$

Therefore

$$t = \int \frac{dz}{(1+z)H} \ .$$

We have seen that

$$H^2 = H_0^2 \left[\Omega_\lambda + (1+z)^3\Omega_m + (1+z)^4\Omega_\gamma\right] \ .$$

Using this, the time of decoupling is obtained as

$$t_d = \frac{1}{H_0}\int_{z_d}^{\infty} \frac{dz'}{(1+z')\sqrt{\Omega_\lambda + \Omega_m(1+z')^3 + \Omega_\gamma(1+z')^4}} \approx 3.7\times 10^5 \text{ years} \ . \quad (4)$$

3. For the abundances we have

$$\Omega_\lambda(z_d) = \left(\frac{\rho_c(z_0)}{\rho_c(z_d)}\right)\Omega_\lambda(z_0), \quad (5)$$

$$\Omega_m(z_d) = \left(\frac{\rho_c(z_0)}{\rho_c(z_d)}\right)\Omega_m(z_0)(1+z_d)^3, \quad (6)$$

$$\Omega_\gamma(z_d) = \left(\frac{\rho_c(z_0)}{\rho_c(z_d)}\right)\Omega_\gamma(z_0)(1+z_d)^4 \ . \quad (7)$$

where $\rho_c(z_0)$ and $\rho_c(z_d)$ correspond to the critical densities today and at the moment of decoupling, respectively. Using that $\rho_c(z_0)/\rho_c(z_d) = H(z_0)^2/H(z_d)^2$ we get

$$\Omega_m(z_d) = \frac{\Omega_m(z_0)(1+z_d)^3}{\Omega_\lambda + (1+z)^3\ \Omega_m + (1+z)^4\ \Omega_\gamma} \ . \quad (8)$$

The expressions for $\Omega_\lambda(z_d)$ and $\Omega_\gamma(z_d)$ are analogous.
When plugging in the numerical values one gets

$$\Omega_\lambda(z_d) = 10^{-9}, \quad \Omega_m(z_d) = 0.774, \quad \Omega_\gamma(z_d) = 0.225. \quad (9)$$

3. Fermion number density

Recall that the number density of particles in cosmic plasma is written as follows,

$$n = g \int f(\vec{p}) \frac{d^3\vec{p}}{(2\pi)^3} = \frac{g}{2\pi^2} \int f(E) E \sqrt{E^2 - m^2} dE, \tag{10}$$

where g is a number of internal degrees of freedom of a particle and $f(\vec{p})$ is the distribution function. To obtain the second equality, one should integrate over angular variables and use the relation

$$E dE = |\vec{p}| d|\vec{p}|, \tag{11}$$

following from the dispersion relation $E = \sqrt{\vec{p}^2 + m^2}$. For light fermions and antifermions we have, correspondingly,

$$n_f = \frac{g}{2\pi^2} \int \frac{dE\ E^2}{e^{\frac{E-\mu}{T}} + 1}, \quad n_{\bar{f}} = \frac{g}{2\pi^2} \int \frac{dE\ E^2}{e^{\frac{E+\mu}{T}} + 1}. \tag{12}$$

Then the difference between n_f and $n_{\bar{f}}$ is given by

$$\Delta n = \frac{g}{2\pi^2} \int_0^\infty dE E^2 \frac{e^{\frac{E+\mu}{T}} - e^{\frac{E-\mu}{T}}}{\left(1 + e^{\frac{E+\mu}{T}}\right)\left(1 + e^{\frac{E-\mu}{T}}\right)}. \tag{13}$$

Expanding with respect to μ/T, we have

$$\Delta n = \frac{g}{\pi^2} \int_0^\infty dE E^2 \left(\frac{\mu}{T} \frac{e^{\frac{E}{T}}}{\left(1 + e^{\frac{E}{T}}\right)^2} + O\left(\frac{\mu^2}{T^2}\right) \right). \tag{14}$$

Integrating by parts the first term, we obtain

$$\Delta n = \frac{2g\mu}{\pi^2} \int_0^\infty \frac{dE \cdot E}{1 + e^{\frac{E}{T}}} = \frac{2g\mu T^2}{\pi^2} \int_0^\infty \frac{dz \cdot z}{e^z + 1} = \frac{g\mu T^2}{6}, \tag{15}$$

where we used the formula

$$\int_0^\infty \frac{z^{2n-1}}{e^z + 1} dz = \frac{2^{2n-1} - 1}{2n} \pi^{2n} B_n, \tag{16}$$

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, ..., \tag{17}$$

valid for positive integer n.