RELATIVITY AND COSMOLOGY II

Solutions to Problem Set 6 29th March 2024

1. Densities evolution

1. The equation governing the evolution of the distribution function $n(t, p^i)$ reads

$$\frac{\partial n}{\partial t} - Hp^i \frac{\partial n}{\partial p^i} = 0 \ , \tag{1}$$

where $i = x, y, z$, and summation over repeated indices is understood. Since the Hubble parameter is $H = \frac{\dot{a}}{a}$, we can rewrite the above as

$$\frac{a}{\dot{a}} \frac{\partial n}{\partial t} - p^i \frac{\partial n}{\partial p^i} = 0 \quad \Leftrightarrow \quad a \frac{\partial n}{\partial a} - p^i \frac{\partial n}{\partial p^i} = 0 \ . \tag{2}$$

2.&3. As customary in the method of characteristics, we assume that $a = a(s)$ and $p^i = p^i(s)$, where s parametrises the characteristic curve. Using the chain rule for the derivatives we obtain

$$\frac{dn}{ds} = \frac{da}{ds} \frac{\partial n}{\partial a} + \frac{dp^i}{ds} \frac{\partial n}{\partial p^i} \ . \tag{3}$$

Comparing the above with (2), we immediately find the following set of Ordinary Differential Equations

$$\frac{da}{ds} = a \quad \Leftrightarrow \quad s = \log\left(\frac{a}{a_0}\right) \ , \tag{4}$$

$$\frac{dp^i}{ds} = -p^i \quad \Leftrightarrow \quad p^i = p_0^i \frac{a_0}{a} \ , \tag{5}$$

where $a_0 = a(0)$ and $p_0^i = p^i(0)$ are the initial conditions.

4. Moreover,

$$\frac{dn}{ds} = 0 \ , \tag{6}$$

which implies that n remains constant along the curve. Therefore, if we assume that initially $n(0, p_0^i) = n_0$, we obtain $n(t, p_0^i) = n\left(t, p^i \frac{a}{a_0}\right)$. This clearly shows that the only effect the expansion of the Universe has on the distribution function of the particles is the redshift in the momentum.

2. Conservation of chemical potential

1. We denote by f_i the distribution function of i's particle participating in the reaction $(1) + (2) \to (3) + (4)$. The vanishing of the collision integral in equilibrium leads to the equality
$$(1 \pm f_1)(1 \pm f_2)f_3 f_4 = (1 \pm f_3)(1 \pm f_4)f_1 f_2 \,, \tag{7}$$
where the upper sign stands for bosons and the lower sign stands for fermions. This equation implies
$$\log \frac{f_1}{1 \pm f_1} + \log \frac{f_2}{1 \pm f_2} = \log \frac{f_3}{1 \pm f_3} + \log \frac{f_4}{1 \pm f_4} \,. \tag{8}$$
We conclude that $\log \frac{f}{1 \pm f}$ is an additive constant of motion. Now we exploit the fact that in equilibrium we know the distribution functions :
$$f_i = \frac{1}{e^{\frac{E_i - \mu_i}{T}} \mp 1} \,, \tag{9}$$
where again the upper sign holds refers to bosons and the lower one to fermions. This is equivalent to :
$$\log \frac{f_i}{1 \pm f_i} = -\frac{E_i - \mu_i}{T} \,. \tag{10}$$
Substituting into eq. (8) and using that energy is conserved, we get
$$\mu_1 + \mu_2 = \mu_3 + \mu_4 \,. \tag{11}$$
2. It is straightforward to generalize the above argument and deduce that
$$\sum_{i=1}^{n} \mu_i = \sum_{i=1}^{m} \bar{\mu}_i \,. \tag{12}$$
3. Consider the reaction $e^+e^- \to e^+e^-\gamma$. From the conservation of the chemical potential it follows immediately that $\mu_\gamma = 0$.
4. Consider the reaction of annihilation, $p + \bar{p} \to 2\gamma$. Since $\mu_\gamma = 0$, we conclude that $\mu_p = -\mu_{\bar{p}}$.

3. Proton gas

1. We saw in the lecture that the average path traveled between two reactions is $\lambda = \frac{1}{\sigma n}$, with $\sigma = m_\pi^{-2}$ the cross section and
$$n = 2 \left(\frac{m_p T}{2\pi} \right)^{3/2} \exp\left(\frac{-m_p}{T} \right)$$
the density of protons. The interaction rate, i.e. the frequency of collisions, is given by
$$\Gamma = \frac{v}{\lambda} = \sigma n v, \tag{13}$$

where v is the average velocity of the protons. It can be evaluated assuming that the kinetic energy of the gas is equal to $\frac{3}{2}T$:

$$v = \sqrt{\frac{3T}{m_p}} \ .$$

Now we compare the rate (13) with the Hubble parameter

$$H = 1.66 g_*^{1/2} \frac{T^2}{M_{Pl}},$$

which sets the rate at which the universe evolves. Decoupling takes places when the two rates are on the same order :

$$1.66 g_*^{1/2} \frac{T_d^2}{M_{Pl}} = \frac{1}{m_\pi^2} \left(\frac{m_p T}{2\pi} \right)^{3/2} \exp\left(\frac{-m_p}{T_d} \right) \sqrt{\frac{3T_d}{m_p}} \, ,$$

where T_d is the decoupling temperature. We get

$$T_d = \frac{m_p}{\ln\left(\frac{\sqrt{3} M_{Pl} m_p}{1.66 g_*^{1/2} (2\pi)^{3/2} m_\pi^2} \right)} = \frac{m_p c^2}{k_B \ln\left(\frac{\sqrt{3} M_{Pl} m_p}{1.66 g_*^{1/2} (2\pi)^{3/2} m_\pi^2} \right)} = 2.5 \cdot 10^{11}\,\mathrm{K}.$$

2. The density ratio is

$$\frac{n_p}{n_\gamma} = \frac{2 \left(\frac{m_p c^2 T_d}{2\pi k_B} \right)^{3/2} \exp\left(\frac{-m_p c^2}{k_B T_d} \right)}{\frac{2\zeta(3)}{\pi^2} T_d^3} = 5.3 \cdot 10^{-18}.$$

3. When $T = T_0 = 2.73\,\mathrm{K}$, the number density of photons is

$$n_\gamma^0 = \frac{2\zeta(3)}{\pi^2} T_0^3 = \frac{2\zeta(3)}{\pi^2} \frac{T_0^3 k_B^3}{c^3 \hbar^3} = 4.2 \cdot 10^8\,\mathrm{m}^{-3}.$$

Assuming that the reactions of annihilation are negligible since decoupling, the density ratio has not changed, so $n_p^0 \simeq 10^{-10}\,\mathrm{m}^{-3}$. Note that this result is extremely sensitive to T. In the real Universe, we have about $n_p^0 = 1$. It is therefore necessary to have a more complicated mechanism to explain the presence of matter in the universe.