RELATIVITY AND COSMOLOGY II

Solutions to Problem Set 5 22nd March 2024

1. Dipole anisotropy of the Cosmic Microwave Background

The distribution function of photons reads as follows,[1]

$$N(p) = \frac{1}{e^{p/T} - 1} \ . \tag{1}$$

Here we denote $p = |\vec{p}|$. The distribution function is a scalar with respect to Lorentz transformations. This means that if we move to a reference frame that is moving with respect to the CMB reference frame, then

$$N_{OBS}(p_{OBS}) = N_{CMB}(p_{CMB}) \ . \tag{2}$$

Here $\vec{p}_{CMB}$ is the momentum of a photon which is at rest in the medium, whereas $\vec{p}_{OBS}$ is the momentum of the photon measured in the Earth. Their absolute values are related by Doppler effect formula,

$$p_{CMB} = \frac{1 - v\cos\theta}{\sqrt{1-v^2}} p_{OBS} \ , \tag{3}$$

where θ is the angle between $\vec{p}_{OBS}$ and the axis along which the Earth is moving with respect to isotropic CMB background. Therefore, from Eqs. (1) and (2) we get

$$\frac{p_{OBS}}{T_{OBS}} = \frac{p_{CMB}}{T_{CMB}}, \tag{4}$$

for any given momentum of the photon. Then, the temperature acquires dependence on θ and is given by

$$T_{OBS} = T_{CMB} \left(\frac{\sqrt{1-v^2}}{1 - v\cos\theta} \right) \ . \tag{5}$$

Expanding the above in powers of v and keeping the first order term, we find

$$T_{OBS} \approx T_{CMB}(1 + v\cos\theta) \quad \to \quad \frac{\delta T}{T} = \frac{T_{OBS} - T_{CMB}}{T_{CMB}} \approx v\cos\theta \ . \tag{6}$$

Considering a dipole anisotropy of the order of 10^{-3}, we find for $\theta = 0$

$$v \approx 10^{-3} \ , \tag{7}$$

or, in conventional units,

$$v \approx 300 \text{ km/s} \ . \tag{8}$$

This order-of-magnitude estimate appears to be quite close to the exact value of the Earth's relative velocity $v \approx 370\,\text{km/s}$.

1. Here and below we use the system of units in which $\hbar = k = c = 1$.

2. Effective number of degrees of freedom

1. For the ultrarelativistic bosons and fermions in thermal equilibrium the energy density is computed by

$$\rho_{f/b,i}(T) = \int_{\mathbb{R}^3} \frac{\mathrm{d}\vec{k}^3}{(2\pi)^3} \rho_{f/b,i}(\vec{k}, T) = g_i \int_{\mathbb{R}^3} \frac{\mathrm{d}\vec{k}^3}{(2\pi)^3} \frac{E_{f/b,i}(\vec{k})}{e^{E_{f/b,i}(\vec{k})/T} \pm 1}. \tag{9}$$

In the relativistic limit $E_{f/b,i}(\vec{k}) \simeq |\vec{k}|$, and the integral can be made adimensional through a change of variable $x = |\vec{k}|/T$:

$$\rho_{f/b,i}(T) = \frac{g_i}{(2\pi)^3} T^4 \int_0^\infty 4\pi x^2 \mathrm{d}x \frac{x}{e^x \pm 1} = \frac{g_i}{2\pi^2} T^4 \int_0^\infty \mathrm{d}x \frac{x^3 e^{-x}}{1 \pm e^{-x}}. \tag{10}$$

The integral can be performed with the following trick :

$$\begin{aligned} \int_0^\infty \mathrm{d}x \frac{x^3 e^{-x}}{1 \pm e^{-x}} &= \int_0^\infty \mathrm{d}x x^3 e^{-x} \sum_{n=0}^{+\infty} (\mp)^n e^{-nx} \\ &= \sum_{n=0}^{+\infty} (\mp)^n \int_0^\infty \mathrm{d}x x^3 e^{-(n+1)x} \\ &= \sum_{n=0}^{+\infty} \frac{(\mp)^n}{(n+1)^4} \underbrace{\int_0^\infty \mathrm{d}y y^3 e^{-y}}_{\Gamma(4)=3!} . \end{aligned} \tag{11}$$

Now, $\sum_{n=0}^{+\infty} \frac{1}{(n+1)^4} = \zeta(4) = \frac{\pi^4}{90}$ [2], while

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{(-)^n}{(n+1)^4} &= \sum_{m=1}^{+\infty} \frac{1}{(2m+1)^4} - \sum_{m=1}^{+\infty} \frac{1}{(2m)^4} \\ &= \sum_{n=1}^{+\infty} \frac{1}{n^4} - 2 \cdot \sum_{n=1}^{+\infty} \frac{1}{(2n)^4} \\ &= \zeta(4) - 2 \cdot \frac{\zeta(4)}{16} = \frac{7}{8} \zeta(4). \end{aligned} \tag{12}$$

Putting it all together we get :

$$\rho_{b,i} = g_{b,i} \frac{\pi^2}{30} T^4, \qquad \rho_{f,j} = \frac{7}{8} g_{f,j} \frac{\pi^2}{30} T^4. \tag{13}$$

Therefore, the total energy density in the radiation dominated epoch is

$$\rho = \sum_i \rho_{b,i} + \sum_j \rho_{f,j} = \left(\sum_i g_{b,i} + \frac{7}{8} \sum_j g_{f,j} \right) \frac{\pi^2}{30} T^4 \equiv g_* \frac{\pi^2}{30} T^4. \tag{14}$$

2. To solve this problem it would be convenient to summarize the particle content of the Standard Model of particle physics. In the table below we list particles sorting them by mass and show the number of degrees of freedom (DOF). We also separated particles that participate and do not participate in strong interaction because of the QCD transition (see details below).

2. If you don't remember the first values of the Riemann ζ function you can always compute them by choosing wisely a periodic function to integrate over its period, and relate it to the sum of its Fourier coefficients.

Particle	Mass (GeV)	Type	DOF
Not strongly interacting particles			
γ	0	boson	2
$\nu_{e,\mu,\tau}$	$\sim 10^{-11}$	fermions	$3 \cdot 2 = 6$
e	$5.11 \cdot 10^{-4}$	fermion	4
μ	0.106	fermion	4
τ	1.78	fermion	4
$W^{\pm}$	80.4	boson	$2 \cdot 3 = 6$
Z^0	91.2	boson	3
h	125	boson	1
Strongly interacting particles			
g	0	bosons	$8 \cdot 2 = 16$
u	$\sim 2 \cdot 10^{-3}$	fermion	$3 \cdot 4 = 12$
d	$\sim 5 \cdot 10^{-3}$	fermion	$3 \cdot 4 = 12$
s	0.095	fermion	$3 \cdot 4 = 12$
c	1.25	fermion	$3 \cdot 4 = 12$
b	4.2	fermion	$3 \cdot 4 = 12$
t	173	fermion	$3 \cdot 4 = 12$

To construct this table you should take into account that each fermion has 4 DOF (two spin states $\times$ particles/antiparticles), neutrinos are special fermions with only one spin state (left-chiral), massless vector bosons have 2 DOF, massive vector bosons have 3 DOF, while Higgs boson h is a scalar boson with only 1 DOF. Also, you should not forget that there are 8 different gluon types (g) and each quark has 3 colors. (For this consideration each color can be treated as an independent particle.) We get :

(a) For $T = 1$ TeV all Standard Model particles are present in plasma and are ultra-relativistic. This gives the total number of bosonic degrees of freedom $g_b = 28$ and fermionic $g_f = 90$, so

$$g_*(1\ \text{TeV}) = 28 + \frac{7}{8} \cdot 90 = 106.75. \tag{15}$$

(b) For $T = 10$ GeV we should exclude too heavy particles ($W^{\pm}$, Z^0, h, t). This reduces number of degrees of freedom to $g_b = 18$ and fermionic $g_f = 78$, so

$$g_*(10\ \text{GeV}) = 18 + \frac{7}{8} \cdot 78 = 86.25. \tag{16}$$

(c) As $T = 10$ MeV is lower than the temperature of QCD transition T_{QCD} there are no free gluons and quarks in the plasma anymore. Moreover, the lightest baryon – pion – has mass ~ 135 MeV, which means that baryons are also not present in the plasma. The particles that contribute at this temperature are photons, neutrinos and electrons which results in

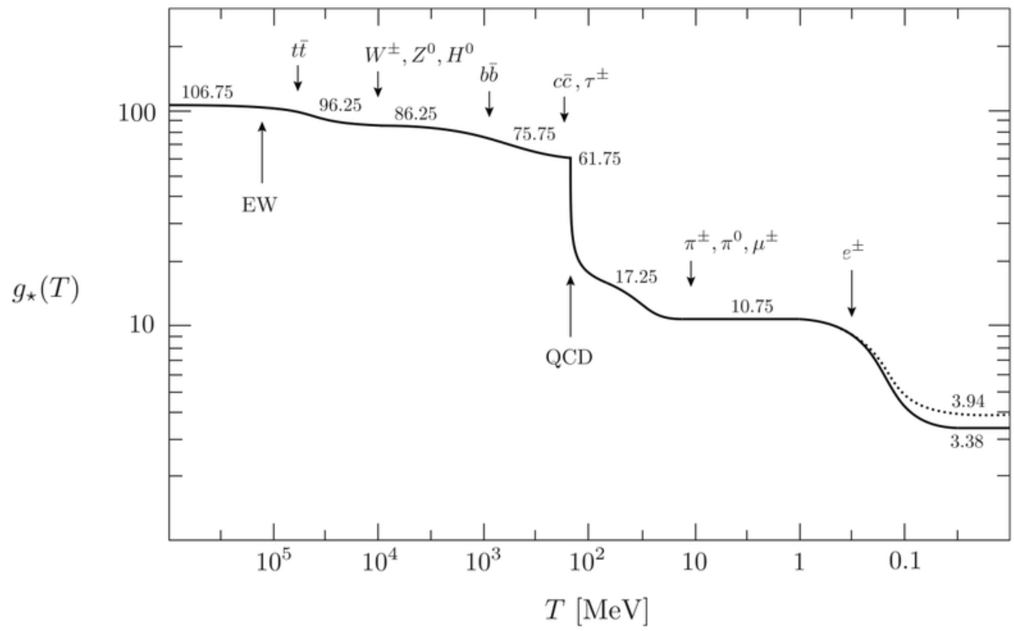
$$g_*(10\ \text{MeV}) = 2 + \frac{7}{8} \cdot 10 = 10.75. \tag{17}$$

(d) At the temperature $T = 0.1$ MeV only photons and neutrinos are present in plasma, which gives

$$g_*(0.1\ \text{MeV}) = 2 + \frac{7}{8} \cdot 6 = 7.25. \tag{18}$$

However, this is a wrong result. The drawback here is in the assumption of the thermal equilibrium. As you will discuss later in the course, neutrinos decouple from the other particles a $T \sim 1$ MeV and at $T = 0.1$ MeV they have temperature that is different from the temperature of the electromagnetic plasma. The correct value of g_* in this case is $g_*(0.1\text{ MeV}) \approx 3.36$.

The evolution of the g_* is shown in the figure below :



3. Thermodynamics of non-relativistic medium

Recall that the distribution function $N(p)$ of gas in equilibrium is given by

$$N(p) = \frac{1}{e^{\frac{E(p)-\mu}{T}} \mp 1}\,, \tag{19}$$

where $E(p) = \sqrt{p^2+m^2}$, $p = |\vec{p}|$, and "$-$" stands for bosons while "$+$" stands for fermions. In the non-relativistic case $E(p) \approx m + \frac{p^2}{2m}$, and under the assumptions $m/T \gg 1$, $(m-\mu)/T \gg 1$ the distribution functions of particles and antiparticles become

$$N_{p,a}(p) = e^{\frac{\mu_{p,a}-m}{T}} e^{-\frac{p^2}{2mT}}\,. \tag{20}$$

Denote $\mu \equiv \mu_p$, then $\mu_a = -\mu$.
Before computing thermodynamical quantities, let us consider here for further convenience the general Gaussian integral of the form

$$\begin{aligned} I_{2n} &= \int_0^{+\infty} e^{-\frac{p^2}{2mT}} p^{2n}\, dp = ||\frac{p^2}{2mT} = y|| = \frac{1}{2}(2mT)^{\frac{2n+1}{2}} \int_0^{+\infty} e^{-y} y^{n-\frac{1}{2}}\, dy = \\ &= \frac{1}{2}(2mT)^{\frac{2n+1}{2}} \Gamma(n+\frac{1}{2}) = (2n-1)!!\sqrt{\frac{\pi}{2}}(mT)^{\frac{2n+1}{2}}\,, \end{aligned} \tag{21}$$

where Γ is the gamma function and $(2n-1)!! = (2n-1)\cdot(2n-3)\cdot\ldots\cdot 3\cdot 1$. We will use it below.

1. The number density of particles in the gas is given by

$$n_p = g\int N_p(p)\frac{d^3\vec{p}}{(2\pi)^3}\,, \tag{22}$$

where g denotes the number of internal degrees of freedom of the particle (or antiparticle). Substituting (20) into (22), we arrive at

$$n_p = \frac{g}{2\pi^2} e^{\frac{\mu-m}{T}} \underbrace{\int_0^{\infty} e^{-\frac{p^2}{2mT}} p^2\, dp}_{I_2=\sqrt{\pi/2}(mT)^{3/2}} = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{\frac{\mu-m}{T}}\,. \tag{23}$$

The number density of antiparticles reads similarly,

$$n_a = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{\frac{-\mu-m}{T}}\,. \tag{24}$$

The total number density is $n = n_p + n_a$.

2. The energy density of particles is, by definition,

$$\rho_p = g\int N_p(p)E(p)\frac{d^3\vec{p}}{(2\pi)^3}\,. \tag{25}$$

To compute this in the non-relativistic limit, we substitute $E(p) \approx m + \frac{p^2}{2m}$ and obtain

$$\rho_p = mn_p + \frac{1}{2m}\frac{g}{2\pi^2} e^{\frac{\mu-m}{T}} \underbrace{\int_0^{\infty} e^{-\frac{p^2}{2mT}} p^4\, dp}_{I_4=3\sqrt{\pi/2}(mT)^{5/2}} = mn_p + \frac{3}{2}n_pT\,. \tag{26}$$

Similarly, for antiparticles $\rho_a = mn_a + \frac{3}{2}n_aT$, and the total energy density is

$$\rho = mn + \frac{3}{2}nT\,. \tag{27}$$

3. The calculation of pressure is slightly more involved. Let us first consider the generic standard definition of the pressure. Consider a small area ΔS placed perpendicular to the direction of z-axis. The number of particles (or antiparticles) with momenta from p to $p+dp$ hitting the area during the time Δt is

$$\Delta n = \frac{1}{2}|v_z|N_{p,a}(p)\frac{d^3\vec{p}}{(2\pi)^3}\Delta S\Delta t\,, \tag{28}$$

where $|v_z| = \frac{|p_z|}{E}$, and 1/2 accounts for the fact that only half of the particles go towards the area. When it hits the surface, the particle transfers to it the momentum $\Delta p_z = 2p_z$. By definition, the pressure is a total momentum transferred to the surface per unit area per unit time, that is

$$P_{p,a} = g\int\frac{\Delta n\Delta p_z}{\Delta S\Delta t} = g\int|v_z|p_zN_{p,a}(p)\frac{d^3\vec{p}}{(2\pi)^3} = g\int\frac{p_z^2}{E}N_{p,a}(p)\frac{d^3\vec{p}}{(2\pi)^3}\,. \tag{29}$$

We would get exactly the same expression if we started with the definition of the stress-energy tensor for an ensemble of point-like particles derived in Lecture 10 in the previous semester :

$$T_{\mu\nu} = g\int\frac{d^3\vec{p}}{(2\pi)^3}\,N(p)\frac{p_\mu p_\nu}{E}. \tag{30}$$

In the isotropic case, $T_{ij} = -\eta_{ij}P$, therefore, e.g., $P = T_{zz}$ which coincides with Eq. (29).

Due to isotropy of the medium we can write $p_z^2 = \frac{1}{3}p^2$, hence

$$P_{p,a} = \frac{g}{6\pi^2}\int_0^\infty\frac{p^4dp}{E(p)}N_{p,a}(p)\,. \tag{31}$$

Calculating this integral in the non-relativistic limit (i.e., $E(p)\approx m$ in the denominator), we get the answer

$$P_{p,a} = \frac{g}{6\pi^2 m}e^{\frac{\pm\mu-m}{T}}\underbrace{\int_0^\infty e^{-\frac{p^2}{2mT}}p^4\,dp}_{I_4=3\sqrt{\pi/2}(mT)^{5/2}} = gT\left(\frac{mT}{2\pi}\right)^{3/2}e^{\frac{\pm\mu-m}{T}} = Tn_{p,a}\,, \tag{32}$$

and, hence, $P = Tn$. Note that $P \ll \rho$, since $T \ll m$. The conclusion is that for the non-relativistic gas one can neglect its pressure and regard the condition $P = 0$ as its equation of state.

4. Let us outline one way to derive the expression for the entropy density. We start from the relations

$$s = \frac{S}{V}\,,\quad \rho = \frac{E}{V}\,,\quad n = \frac{N}{V}\,. \tag{33}$$

Taking the differentials, we have

$$\begin{aligned} dE &= Vd\rho + \rho dV\,, \\ dN &= Vdn + ndV\,, \\ dS &= Vds + sdV\,. \end{aligned} \tag{34}$$

Next, we recall the first law of thermodynamics :

$$dE = TdS - PdV + \sum_i \mu_i dN_i\,, \tag{35}$$

where the sum runs over all species present in the system. Substituting (34) into (35), we have

$$\left(Ts - P - \rho + \sum_i \mu_i n_i\right) dV + \left(Tds - d\rho + \sum_i \mu_i dn_i\right) V = 0\,. \tag{36}$$

This equation is valid for the whole system as well as for any part of it. In particular, we can take some region of a constant volume, $dV = 0$, and obtain

$$Tds - d\rho + \sum_i \mu_i dn_i = 0\,. \tag{37}$$

Plugging this back, we get the desired expression for the entropy density,

$$s = \frac{P + \rho - \sum_i \mu_i n_i}{T}\,. \tag{38}$$

Using (27) and (32), one can write this as

$$s = \frac{5}{2}n + \frac{m}{T}n - \frac{1}{T}\sum_i \mu_i n_i\,, \tag{39}$$

where n_i are given by (23) and (24).
Alternatively, we can use the following equation given in the lecture,

$$s = -g\int [N(p)\log N(p) \mp (1 \pm N(p))\log(1 \pm N(p))]\frac{d^3\vec{p}}{(2\pi)^3}, \tag{40}$$

where the upper sign refers to bosons and the lower sign to fermions. In the non-relativistic limit, we can approximate the second part

$$\mp(1 \pm N)\log(1 \pm N) \sim \mp(1 \pm N)[\pm N - \frac{(\pm N)^2}{2} + \cdots] \sim -N \tag{41}$$

and get

$$\begin{aligned} s = -g\sum_{i=p,a}\int \frac{d^3\vec{p}}{(2\pi)^3}N_i(p)[\log N_i(p) - 1] = -g\sum_{i=p,a}\int \frac{d^3\vec{p}}{(2\pi)^3}N_i(p)[\frac{\mu_i}{T} - \frac{E(p)}{T} - 1] = \\ = -\frac{1}{T}\sum_i \mu_i n_i + \frac{\rho}{T} + n, \end{aligned} \tag{42}$$

which coincides with the result in Eq. (39).