RELATIVITY AND COSMOLOGY II

Solutions to Problem Set 4 15th March 2024

1. Measuring dark energy by luminosity distance

1&2. As in both parts of the problem abundances sum up to 1 ($\Omega_M + \Omega_\Lambda$ and $\Omega_M + \Omega_{\rm DE}$), the Universe is flat. The energy conservation for matter X with arbitrary equation of state is

$$\partial_t(\rho_X R^3) + p_X \partial_t(R^3) = 0, \tag{1}$$

and solving that for arbitrary $p_X = \omega \rho_X$ for solutions of form $\rho_X \propto R^\alpha$ gives equation

$$(3+\alpha)R^{\alpha+2}\dot{R} + 3w \cdot R^{\alpha+2}\dot{R} = 0. \tag{2}$$

That is solved if $\alpha = -3(w+1)$. That gives some already known results and a new one for dark energy:

$$\begin{aligned} w_M &= 0, & \rho_M &\approx R^{-3}, && \text{(Dust)} \\ w_R &= \frac{1}{3}, & \rho_R &\approx R^{-4}, && \text{(Radiation)} \\ w_\Lambda &= -1, & \rho_\Lambda &\approx 1, && \text{(Cosmological constant)} \\ w_{\rm DE} &= -0.9, & \rho_{\rm DE} &\approx R^{-\frac{3}{10}}. && \text{(Dark energy)} \end{aligned}$$

In terms of Friedman equation the time evolution of the universe with matter end either cosmological constant or the proposed dark energy model is given

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2 \left[\Omega_M \left(\frac{R_0}{R}\right)^3 + \Omega_\Lambda\right], \tag{3}$$

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2 \left[\Omega_M \left(\frac{R_0}{R}\right)^3 + \Omega_{\rm DE} \left(\frac{R_0}{R}\right)^{\frac{3}{10}}\right]. \tag{4}$$

As seen in the lecture, this can be put into a useful form for computing the redshift via defining $x = R_0/R$ and $A = \dot{R}/(RH_0)$, so

$$A_\Lambda^2 = \Omega_M x^{-3} + \Omega_\Lambda, \tag{5}$$

$$A_{\rm DE}^2 = \Omega_M x^{-3} + \Omega_{\rm DE} x^{-0.3}, \tag{6}$$

with a nice relationship to the luminosity distance

$$d_L(z) = \frac{1+z}{H_0} \int_{\frac{1}{1+z}}^1 \frac{dx}{x^2 A}. \tag{7}$$

This leads to final formulas of

$$d_{L,\Lambda}(z) = \frac{1+z}{H_0} \int_{\frac{1}{1+z}}^1 \frac{dx}{\sqrt{\Omega_M x + \Omega_\Lambda x^4}}, \tag{8}$$

$$d_{L,\rm DE}(z) = \frac{1+z}{H_0} \int_{\frac{1}{1+z}}^1 \frac{dx}{\sqrt{\Omega_M x + \Omega_{\rm DE} x^{3.7}}}. \tag{9}$$

3. The elegant square roots of polynomials under the integral will melt down either your brain or your computer if they're tried to solved analytically, so to get some estimations one must resort to numerics. Numerical integration for given abundances gives answers of

$$d_{L,\Lambda}(z=1) \approxeq \frac{1.5329}{H_0}, \tag{10}$$

$$d_{L,\mathrm{DE}}(z=1) \approxeq \frac{1.5115}{H_0}, \tag{11}$$

so the precision one needs to distinguish between these two models is

$$\frac{|d_{L,\Lambda}(z=1) - d_{L,\mathrm{DE}}(z=1)|}{d_{L,\Lambda}(z=1)} \approxeq 0.0203. \tag{12}$$

4. The formula derived in the previous problem sheet

$$d_L = \frac{z}{H_0} + \frac{1}{2H_0}(1-q_0)z^2 \tag{13}$$

gives the second order correction to luminosity distance in terms of deceleration parameter $q_0 = -\frac{1}{H_0^2 R_0}\ddot{R}_0$. Consider Freedman equation with arbitrary matter content

$$(\dot{R})^2 = H_0^2 R^2 \left[\sum \Omega_X \left(\frac{R_0}{R}\right)^{3(w_X+1)}\right]. \tag{14}$$

Taking a derivative with respect to t gives

$$2\dot{R}\ddot{R} = H_0^2 \left[\sum -(3w_X+1)\Omega_X \frac{R_0^{3w_X+3}}{R^{3w_X+2}}\dot{R}\right]. \tag{15}$$

Simplifying and evaluating at R_0 gives

$$q_0 = \frac{1}{2}\sum(3w_x+1)\Omega_X. \tag{16}$$

For matter content considered in the exercise one has

$$q_{\mathrm{DE}} = \frac{1}{2}(0.3 - 0.7\cdot 1.7), \tag{17}$$

$$q_\Lambda = \frac{1}{2}(0.3 - 0.7\cdot 2), \tag{18}$$

and formula for difference in luminosity distance

$$\frac{|d_{L,\Lambda}(z=1) - d_{L,\mathrm{DE}}(z=1)|}{d_{L,\Lambda}(z=1)} = \frac{\frac{1}{2}(q_{\mathrm{DE}} - q_\Lambda)}{1+\frac{1}{2}(1-q_\Lambda)} = \frac{0.0525}{1.775} \approx 0.0296\ . \tag{19}$$

2. Fate of the Universe

1. We start from the first Friedmann equation written using abundances

$$\Omega_M + \Omega_k + \Omega_\Lambda = 1.$$

The line $\Omega_\Lambda = 1 - \Omega_M$ defines a flat universe. Above this curve, we have $k = 1$ and $k = -1$ below. To study acceleration, we subtract the second Friedmann equation to obtain

$$\frac{\ddot{R}}{R} + \frac{4\pi G}{3}\rho - \frac{\Lambda}{3} = 0.$$

By identifying the abundances, we get

$$\Omega_\Lambda = \frac{\Omega_M}{2} + \frac{\ddot{R}}{RH^2}.$$

The curve $\Omega_\Lambda = \Omega_M/2$ describes a universe with a zero acceleration. So we have a universe in acceleration above the curve and deceleration below.

2. Writing Friedman equations in term of abundances:

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\Omega_\Lambda + \Omega_k\left(\frac{R_0}{R}\right)^2\right). \tag{20}$$

The equation implies the contents of the brackets on the right hand side is positive.

$$\Omega_\Lambda R^2 + (1 - \Omega_\Lambda){R_0}^2 > 0. \tag{21}$$

This equation excludes initial singularity $R = 0$ for $\Omega_\Lambda > 1$.

For $\Omega_\Lambda < 1$ the expansion rate $\dot{R}$ is bounded from below, therefore singularity must have occurred in the past.

3. For the case where we neglect the cosmological constant, we can write the first Friedmann equation:

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\Omega_k\left(\frac{R_0}{R}\right)^2 + \Omega_M\left(\frac{R_0}{R}\right)^3\right).$$

Swapping variables to $r = \frac{R}{R_0}$, $\dot{r} = \frac{\dot{R}}{R_0}$ yields

$$\dot{r}^2 = \Omega_M\frac{1}{r} - \Omega_k\ .$$

As $\Omega_M + \Omega_k = 1$, we get

$$\dot{r}^2 = H_0^2\left[\Omega_M\left(\frac{1}{r} - 1\right) + 1\right].$$

We can write this equation as:

$$E_{kin} + U(r) = E_{tot},$$

with

$$U(r) = -\frac{H_0^2 \Omega_M^0}{r},$$

and

$$E_{tot} = H_0^2 \left(1 - \Omega_M^0\right).$$

The potential is monotonic, negative and $U(r) \to 0$, when $r \to \infty$. Then, we first deduce, that for all possible values of E_{tot}, there is an initial singularity. Secondly, depending on the sign of E_{tot}, we have an infinite expansion or a collapse in the future. For $E_{tot} > 0$, we have an infinite expansion. Then, $\Omega_M^0 < 1$ is the condition to have an infinite expansion. Otherwise, the universe eventually collapses in the future.

4. For this point, we proceed as before. The Friedman equation, including both matter and Λ is

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2 \left(\Omega_\Lambda \left(\frac{R_0}{R}\right)^0 + \Omega_k \left(\frac{R_0}{R}\right)^2 + \Omega_M \left(\frac{R_0}{R}\right)^3\right),$$

and substituting $r = R/R_0$, $\Omega_k = 1 - \Omega_M - \Omega_\Lambda$ gives

$$E_{kin} + U(r) = 0,$$

with

$$U(r) = -H_0^2 \left[\Omega_M \left(\frac{1}{r} - 1\right) + \Omega_\Lambda \left(r^2 - 1\right) + 1\right].$$

We will study this potential in the cases where $\Omega_\Lambda > 0$ and $\Omega_\Lambda < 0$ separately. Case $\Omega_\Lambda < 0$: First, remark that $U(r)$ goes from $-\infty$, for small r, to ∞ for large r. We compute the derivative:

$$U'(r) = -\frac{H_0^2}{2} \left[2\Omega_\Lambda^0 r - \frac{\Omega_M^0}{r^2}\right] > 0.$$

The potential is monotone over the region $r > 0$. Then, there is an initial singularity and the universe grows until r_b, which is define as $U(r_b) = 0$, then it decreases and collapse.

Case $\Omega_\Lambda > 0$: In this case, $U(r) \to -\infty$ for both $r \to 0$ and $r \to \infty$. Then, the potential has a maximum, given by

$$U'(r_{max}) = -\frac{H_0^2}{2} \left[2\Omega_\Lambda r_{max} - \frac{\Omega_M}{r_{max}^2}\right] = 0\ , \tag{22}$$

so

$$r_{max} = \left(\frac{\Omega_M}{2\Omega_\Lambda}\right)^{1/3}. \tag{23}$$

And we get:

$$U_{max} = U(r_{max}) = -H_0^2 \left[\frac{3}{2^{2/3}} (\Omega_M)^{2/3} (\Omega_\Lambda)^{1/3} + (1 - \Omega_M - \Omega_\Lambda)\right]. \tag{24}$$

Depending on the sign of U_{max}, we have no zero or two zeros if $U_{max} < 0$ or $U_{max} > 0$ respectively. To study this inequalities, we find the solutions of the equation $U_{max} =$

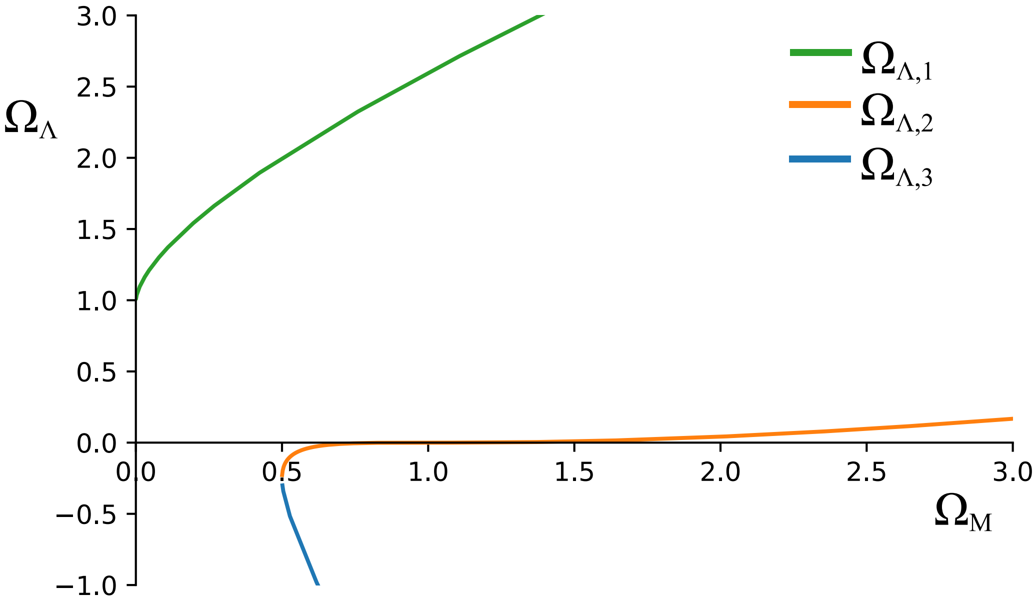
0. They are given by

$$\begin{cases} \Omega_{\Lambda,1} = 1 - \Omega_M + \frac{3}{2}\left(\frac{\Omega_M^2}{D^{1/3}} + D^{1/3}\right), \\ \Omega_{\Lambda,2} = 1 - \Omega_M - \frac{3}{2}\left(\frac{\Omega_M^2}{D^{1/3}}e^{\frac{i\pi}{3}} + D^{1/3}e^{\frac{-i\pi}{3}}\right), \\ \Omega_{\Lambda,3} = 1 - \Omega_M - \frac{3}{2}\left(\frac{\Omega_M^2}{D^{1/3}}e^{\frac{-i\pi}{3}} + D^{1/3}e^{\frac{i\pi}{3}}\right) \end{cases}$$

with

$$D = \Omega_M^2 - \Omega_M^3 + \sqrt{\Omega_M^4 - 2\Omega_M^5}.$$

The plot below shows real-valued $\Omega_{\Lambda,i}$'s:



Each of those curves should be interpreted separately. The curve $\Omega_{\Lambda,1}$ lies above line $\Omega_M = 2\Omega_\Lambda$. From (23) one can conclude it corresponds to $r_{max} < 1$ which (as the current moment of evolution of the universe is $r = 1$, by definition) implies the system is prohibited from going from $r = 1$ to $r = 0$, so **lack of singularity in the past** is implied.

The curves $\Omega_{\Lambda,2}, \Omega_{\Lambda,3}$ have complex values for $\Omega_M < \frac{1}{2}$. The curve $\Omega_{\Lambda,2}$ has negative values for $0.5 < r < 1$, and $\Omega_{\Lambda,3}$ is below zero for any of it's real values. The complex values correspond to no real roots of $U_{max} = 0$, which has no physical significance. $\Omega_\Lambda < 0$ correspond to $r_{max} < 0$, which are also unimportant for this problem.

The region of real, positive $\Omega_{\Lambda,2}$ is a space of $U_{max}(r_{max}) = 0$ and $r_{max} > 1$. That means below this curve separates regions where r is bounded from above ($U_{max} > 0$) and not, which corresponds to regions of **eventual recollapse** ($\Omega_\Lambda < \Omega_{\Lambda,2}$) and **infinite expansion** ($\Omega_\Lambda > \Omega_{\Lambda,2}$) respectively.

Remark: Experimentally, it has been measured:

$$\begin{aligned} \Omega_M &= 0.24 \pm 0.04, \\ \Omega_\Lambda &= 0.76 \pm 0.06, \\ \Omega_{total} &= 1.003 \pm 0.017. \end{aligned}$$

The universe has an initial singularity, it will expand forever and is accelerating. It is not known, whether the universe is closed or open, $\Omega_k = -0.003 \pm 0.017$.