RELATIVITY AND COSMOLOGY II

Problem Set 3 8th March 2024

1. Evolution of the universe

1. (a) *Universe dominated by radiation* ($p = \rho/3, \Lambda = k = 0$).
The Friedmann equations become

$$\frac{\dot{R}^2}{R^2} = 8\pi G p \ , \tag{1}$$

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} = -8\pi G p \ . \tag{2}$$

Substituting p from (1) in (2) and multiplying by R^2 gives :

$$\ddot{R}R + \dot{R}\dot{R} = 0 \tag{3}$$

The equation is homogeneous, so good place to start is polynomial solution $R = R_0(\frac{t}{t_0})^a$, and substitution leads to

$$a(a-1)t^{2a-2} + a^2 t^{2a-2} = 0 \ . \tag{4}$$

This leads to $2a = 1$, which leaves us with

$$R(t) = R_0 \left(\frac{t}{t_0}\right)^{1/2} \ . \tag{5}$$

(b) *Universe dominated by the positive cosmological constant*, $p = \rho = k = 0$.
The Friedmann equations become

$$\frac{\dot{R}^2}{R^2} - \frac{\Lambda}{3} = 0 \ , \tag{6}$$

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} - \Lambda = 0. \tag{7}$$

Reducing $\frac{\dot{R}^2}{R^2}$ between equations (6) and (7) leaves us with with

$$\ddot{R} - \frac{\Lambda R}{3} = 0 \ , \tag{8}$$

with solution

$$R(t) = R_0 \exp\left(\sqrt{\frac{\Lambda}{3}}\, t\right) = R_0 \exp(Ht) \ . \tag{9}$$

Note: Such a universe in which only a positive cosmological constant exists is called de Sitter spacetime.

2. *Universe dominated by a negative cosmological constant*, $p = \rho = 0$.
   The Friedmann equations can now be written as

$$\dot{R}^2 + k = \frac{\Lambda}{3} R^2 \, , \tag{10}$$

$$\frac{\ddot{R}}{R} = \frac{\Lambda}{3} \, . \tag{11}$$

The general solution of the second equation is

$$R(t) = A \cos\left(\sqrt{\frac{|\Lambda|}{3}} t\right) + B \sin\left(\sqrt{\frac{|\Lambda|}{3}} t\right) \, .$$

Substituting this into the first equation gives

$$k = \frac{\Lambda}{3}(A^2 + B^2) \, .$$

Since $\Lambda < 0$, we must have $k = -1$, i.e., the universe has negative curvature. With the initial condition $R(0) = 0$, the solution becomes

$$R(t) = \sqrt{\frac{3}{|\Lambda|}} \sin\left(\sqrt{\frac{|\Lambda|}{3}} t\right) \, .$$

Note: Such a universe in which only a negative cosmological constant exists is called anti-de Sitter spacetime.

2. Age of the universe

For the age of matter-dominated flat universe ($p = k = \Lambda = 0$), the Friedman equations are

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3} \rho \, , \tag{12}$$

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} = -8\pi G p \, . \tag{13}$$

Since $p = 0$, the first second equation yields

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} = 0 \, , \tag{14}$$

which (again, unsurprisingly) has power-function solution :

$$R(t) = R_0 \left(\frac{t}{t_0}\right)^{2/3} \, . \tag{15}$$

With the Hubble constant today

$$H_0 = \frac{\dot{R}(t_0)}{R(t_0)} = \frac{2}{3t_0} \, ,$$

we have $t_0 = \frac{2}{3H_0}$. By using $H_0 \approx 70 \frac{\text{km}}{\text{s Mpc}}$ and 1 pc $= 3.086 \times 10^{16}$m, we find

$$t_0 = 2.93 \times 10^{17} \text{ s} = 9.3 \times 10^9 \text{ years} .$$

3. Stages of evolution

To determine the temporal evolution of the scale factor $R(t)$ we consider the Friedmann equations in the form :

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho,$$

$$2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 - \Lambda = -8\pi G p.$$

1. (a) *Flat universe composed of radiation and a cosmological constant* $\Lambda > 0$

   The Friedmann equations become

   $$\left(\frac{\dot{R}}{R}\right)^2 - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho$$

   $$2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 - \Lambda = -\frac{8\pi G}{3}\rho.$$

   We sum the two equations and we obtain the differential equation for $R(t)$.

   $$\ddot{R}R + \dot{R}^2 = \frac{2\Lambda}{3}R^2 .$$

   We can write it as

   $$\frac{1}{2}\frac{d^2}{dt^2}\left(R^2\right) = \frac{2\Lambda}{3}R^2 .$$

   The solution is

   $$R(t)^2 = A^2 \sinh\left(\sqrt{\frac{4\Lambda}{3}}\, t\right) , \tag{16}$$

   where we imposed the boundary condition $R(0) = 0$.

   (b) *Flat universe composed of matter and a cosmological constant* $\Lambda > 0$

   As before we combine the Friedmann equations to find a differential equation for $R(t)$.

   $$\ddot{R}R + \frac{1}{2}\dot{R}^2 = \frac{\Lambda}{2}R^2 .$$

   To solve it, we can change of variables $R = x^\alpha$. The equation become

   $$\ddot{x} + \frac{\dot{x}^2}{x}\left(\frac{3}{2}\alpha - 1\right) = \frac{\Lambda}{2\alpha}x. \tag{17}$$

If we choose $\alpha = \frac{2}{3}$, the second term disappear and we find the following equation :

$$\ddot{x} = \frac{3\Lambda}{4} x \ .$$

Again using the boundary condition $R(0) = 0$, we get the solution

$$x(t) = A \sinh \left(\sqrt{\frac{3\Lambda}{4}}\, t \right) \ .$$

Substituting back, the final result is :

$$R(t) = A^{\frac{2}{3}} \sinh^{\frac{2}{3}} \left(\frac{3}{2} \sqrt{\frac{\Lambda}{3}}\, t \right) \ . \tag{18}$$

Note: At the current moment in cosmological history, our Universe is dominated by non-relativistic matter and an energy content, the properties of which are very similar to those of a cosmological constant. So the solution we just found describes well our current Universe.

2. We can find the age of the universe in function of H_0 and Ω_m. Using (18) we find the Hubble constant at present time t_0 :

$$H_0 = \frac{\dot{R}(t_0)}{R(t_0)} = \sqrt{\frac{\Lambda}{3}} \coth \left(\frac{3}{2} \sqrt{\frac{\Lambda}{3}}\, t_0 \right) \ . \tag{19}$$

By inverting this expression [1], we get :

$$t_0 = \frac{2}{3} \sqrt{\frac{3}{\Lambda}} \text{arccoth} \left(\sqrt{\frac{3H_0^2}{\Lambda}} \right) = \frac{1}{3} \sqrt{\frac{3}{\Lambda}} \ln \frac{\sqrt{\frac{3H_0^2}{\Lambda}} + 1}{\sqrt{\frac{3H_0^2}{\Lambda}} - 1} = \frac{1}{3} \sqrt{\frac{3}{\Lambda}} \ln \frac{1 + \sqrt{\frac{\Lambda}{3H_0^2}}}{1 - \sqrt{\frac{\Lambda}{3H_0^2}}} \ .$$

We can substitute Λ with Ω_m. For this, use the first Friedmann equation at present time t_0.

$$H_0^2 = \frac{\Lambda}{3} + \frac{8\pi G}{3} \rho(t_0) \ . \tag{20}$$

Dividing by H_0^2 we find :

$$\frac{\Lambda}{3H_0^2} = 1 - \Omega_m \ .$$

We can express t_0 as :

$$\begin{aligned} t_0 &= \frac{1}{3H_0} \frac{1}{\sqrt{1-\Omega_m}} \ln \frac{1 + \sqrt{1-\Omega_m}}{1 - \sqrt{1-\Omega_m}} \\ &= \frac{1}{3H_0} \frac{1}{\sqrt{1-\Omega_m}} \ln \frac{(1 + \sqrt{1-\Omega_m})^2}{\Omega_m} \\ &= \frac{2}{3H_0} \frac{1}{\sqrt{1-\Omega_m}} \ln \frac{1 + \sqrt{1-\Omega_m}}{\sqrt{\Omega_m}} \ . \end{aligned}$$

1. We use the following relation $\text{arccoth} x = \frac{1}{2} \ln \frac{x+1}{x-1}$ for $|x| > 1$.

3. Since $\sinh(x) \approx x$ for small x, the approximation of (16) for early times $t^2 \ll \frac{3}{4\Lambda}$ yields

$$R(t)^2 \approx \sqrt{\frac{4\Lambda}{3}} A^2 t\,, \tag{21}$$

which under redefinition of $A^2 = \sqrt{\frac{3}{4\Lambda}} R_0^2/t_0$ gives the answer

$$R(t) \approx R_0 \left(\frac{t}{t_0}\right)^{\frac{1}{2}}\,. \tag{22}$$

This coincides with the previous result (5).
Similarly eq. (18) gives for small times :

$$R(t) \approx A^{\frac{2}{3}} \left(\frac{3}{2}\sqrt{\frac{\Lambda}{3}}\,t\right)^{\frac{2}{3}}\,, \tag{23}$$

which again under suitable redefinition of A corresponds to (15).
To get the correspondence in age of the universe, one needs nothing more than analyzing (19). As $\coth x \approx x^{-1}$, $H_0 \approx \frac{2}{3t_0}$, which is the solution from problem 2.

4. As $\sinh x = \frac{1}{2}(e^x - e^{-x}) \approx e^x$ for big values of x, large times in both cases lead to

$$R(t) \approx \exp\sqrt{\frac{\Lambda}{3}}t\,. \tag{24}$$

This coincides with eq. (9) as derived in 1b.
Note: This computation shows that (according to our current understanding of cosmology) our Universe will be dominated by a cosmological constant in the future. In other words, de Sitter spacetime appears to be the future attractor of our Universe.

4. First correction to the Hubble law

Starting with expansion of propagation distance :

$$\begin{aligned}\bar{r}_1 + O(\bar{r}^3) = \int_{t_1}^{t_0} \frac{dt}{R(t)} &= \int_{t_1}^{t_0} \left(\frac{1}{R(t_0)} - \frac{\dot{R}(t_0)}{R(t_0)^2}(t-t_0) + O(t-t_0)^2\right) = \\ &= \frac{1}{R(t_0)}(t_0 - t_1) + \frac{\dot{R}(t_0)}{2R(t_0)^2}(t_0-t_1)^2 + O((t_0-t_1)^3) = \\ &= \frac{1}{R(t_0)}\Delta t + \frac{H_0}{2R(t_0)}\Delta t^2 + O(\Delta t^3)\end{aligned} \tag{25}$$

with Δt defined as $t_0 - t_1$ for convenience. Note that integrand was only expanded up to linear order to give answer at quadratic. We won't get away with that anymore.
Now expanding $\frac{1}{R(t_1)}$, which occurs both at formula for d and z :

$$\begin{aligned}\frac{1}{R(t_1)} &= \frac{1}{R(t_0)} - \frac{\dot{R}(t_0)}{R(t_0)^2}(t_1-t_0) + \frac{1}{2}\left(\frac{2\dot{R}^2 - \ddot{R}R}{R^3}\right)(t_1-t_0)^2 + O\big((t_1-t_0)^3\big) = \\ &= \frac{1}{R(t_0)}\left(1 + H_0\Delta t + H_0^2(1+\frac{q_0}{2})\Delta t^2\right) + O(\Delta t^3).\end{aligned} \tag{26}$$

Plugging that into redshift :

$$1+z = \left(1 + H_0\Delta t + H_0^2(1+\frac{q_0}{2})\Delta t^2\right) + O(\Delta t^3) \tag{27}$$

$$z = H_0\Delta t + H_0^2(1+\frac{q_0}{2})\Delta t^2 + O(\Delta t^3)\ , \tag{28}$$

and, along with $\bar{r}$ into d :

$$\begin{aligned} d &= R(t_0)^2 \underbrace{\left(\frac{1}{R(t_0)}\Delta t + \frac{H_0}{2R(t_0)}\Delta t^2\right)}_{\bar{r}} \underbrace{\frac{1}{R(t_0)}\left(1 + H_0\Delta t + H_0^2(1+\frac{q_0}{2})\Delta t^2\right)}_{1/R(t_1)} + O(\Delta t^3) = \\ &= \Delta t\left(1 + \frac{H_0}{2}\Delta t\right)\left(1 + H_0\Delta t + H_0^2(1+\frac{q_0}{2})\Delta t^2\right) + O(\Delta t^3) \\ &= \Delta t + \frac{3}{2}H_0\Delta t^2 + O(\Delta t^3) \end{aligned} \tag{29}$$

Last part is reducing Δt between formulas.

$$\begin{aligned} d - \frac{1}{H_0}z &= \frac{3}{2}H_0\Delta t^2 - H_0(1+\frac{q_0}{2})\Delta t^2 + O(\Delta t^3) \\ d - \frac{1}{H_0}z &= \frac{H_0}{2}(1-q_0)\Delta t^2 + O(\Delta t^3) \end{aligned} \tag{30}$$

Now removing Δt^2

$$d - \frac{1}{H_0}z - \frac{1}{2H_0}(1-q_0)z^2 = O(\Delta t^3) \tag{31}$$

The expansion z implies that order of expansion in z matches order of expansion in Δt, so $O(\Delta t^3) = O(z^3)$. Therefore

$$d = \frac{1}{H_0}z + \frac{1}{2H_0}(1-q_0)z^2 + O(z^3) \tag{32}$$

or in more concise (and less precise) notation

$$d = \frac{1}{H_0}\left(z + \frac{1}{2}(1-q_0)z^2\right)\ . \tag{33}$$