RELATIVITY AND COSMOLOGY II

Solutions to Problem Set 2 March 1st, 2024

1. Energy-momentum tensor for a perfect fluid

1. A perfect fluid is an idealized medium with no heat conduction, zero viscosity (no shear forces) and isotropic pressure. This means that at every point, there is a locally inertial frame of reference moving with the fluid. In this comoving frame, the fluid appears the same in all directions.

   Rotational invariance then tells us that $T^{0i} = 0$, since these terms transform as vectors under spatial rotations. Also $T^{ij} = 0$ for $i \neq j$ (these are called "shear terms). So in the rest frame, the energy-momentum tensor of a perfect fluid is diagonal. Lastly, of course the pressure has to be identical in all directions. We are just left with

   $$T^{\mu\nu} = \mathrm{Diag}\left[\rho, p, p, p\right] . \tag{1}$$

   With some trial and error this can be written in a covariant way as

   $$T^{\mu\nu} = (\rho + p)\; u^\mu u^\nu + p\; \eta^{\mu\nu} \;, \tag{2}$$

   where u^μ is the four-velocity of the medium, which in the rest frame reads $u^\mu = (1, 0, 0, 0)$.

   Note: Since the above expression is covariant, it can be immediately generalized to an arbitrary frame by replacing $\eta^{\mu\nu}$ by a general metric $g^{\mu\nu}$:

   $$T^{\mu\nu} = (\rho + p)\; u^\mu u^\nu + p\; g^{\mu\nu} \;. \tag{3}$$

2. In order to facilitate the calculations involved in this part of the exercise, it is convenient to reinstate factors of c. Since ρ is actually a mass density, with units kilogram per cubic metre, and p is a pressure, with units Newton per square metre, the energy-momentum tensor should read

   $$T^{\mu\nu} = (\rho + \frac{p}{c^2})u^\mu u^\nu + p\eta^{\mu\nu} \;. \tag{4}$$

   (a) We wish to compute the following quantity

   $$u^\nu \partial^\mu T_{\mu\nu} \;. \tag{5}$$

   Plugging the explicit form of $T_{\mu\nu}$ into the above, we obtain

   $$\begin{aligned} u^\nu \partial^\mu T_{\mu\nu} &= u^\nu \partial^\mu \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu + u^\nu \left(\rho + \frac{p}{c^2}\right) \partial^\mu (u_\mu u_\nu) + u^\mu \partial_\mu p \\ &= -c^2 \left(u^\mu \partial_\mu \rho + \left(\rho + \frac{p}{c^2}\right) \partial_\mu u^\mu \right) \;. \end{aligned} \tag{6}$$

   Here we used that $u_\mu u^\mu = -c^2$, which implies $u^\mu \partial^\nu u_\mu = 0$. Now, due to conservation of the energy-momentum tensor ($\partial^\mu T_{\mu\nu} = 0$), the above leads to

   $$u^\mu \partial_\mu \rho + \left(\rho + \frac{p}{c^2}\right) \partial_\mu u^\mu = 0 \;. \tag{7}$$

(b) To proceed with the attainment of the non-relativistic limit of the above expression, we first plug in the definition of the four velocity, $u^\mu = (c\gamma, \gamma\vec{u})$, to find

$$\gamma\frac{\partial\rho}{\partial t} + \gamma\vec{u}\cdot\vec{\nabla}\rho + \left(\rho + \frac{p}{c^2}\right)\frac{\partial\gamma}{\partial t} + \left(\rho + \frac{p}{c^2}\right)\vec{\nabla}\cdot(\gamma\vec{u}) = 0\ . \tag{8}$$

Now, for $c \to \infty$ and $\gamma \approx 1$, we obtain the standard continuity equation (one of the Navier-Stokes equations):

$$\frac{\partial\rho}{\partial t} + \vec{u}\cdot\vec{\nabla}\rho + \rho\vec{\nabla}\cdot\vec{u} = 0\ . \tag{9}$$

(c) The conservation law $\partial^\mu T_{\mu\nu} = 0$ reads

$$\partial^\mu\left(\rho + \frac{p}{c^2}\right)u_\mu u_\nu + \left(\rho + \frac{p}{c^2}\right)\partial^\mu(u_\mu u_\nu) + \partial_\nu p = 0\ . \tag{10}$$

Plugging the result of point (a) into the above expression, we find

$$\rho u^\mu\partial_\mu u_\nu + \frac{1}{c^2}u^\mu\partial_\mu(pu_\nu) + \partial_\nu p = 0\ . \tag{11}$$

Consider the spatial part of this expression, i.e. $\nu = i$:

$$\rho u^\mu\partial_\mu(\gamma\vec{u}) + \frac{1}{c^2}u^\mu\partial_\mu(p\gamma\vec{u}) + \vec{\nabla}p = 0\ . \tag{12}$$

Expanding as before, we obtain

$$\rho\gamma\left[\frac{\partial}{\partial t}(\gamma\vec{u}) + (\vec{u}\cdot\vec{\nabla})(\gamma\vec{u})\right] + \frac{\gamma}{c^2}\left[\frac{\partial}{\partial t}(p\gamma\vec{u}) + (\vec{u}\cdot\vec{\nabla})(p\gamma\vec{u})\right] + \vec{\nabla}p = 0\ . \tag{13}$$

In the non-relativistic limit, we see that the above expression reduces to Euler's equation

$$\rho\frac{\partial\vec{u}}{\partial t} + \rho(\vec{u}\cdot\vec{\nabla})\vec{u} + \vec{\nabla}p = 0\ . \tag{14}$$

2. General equation of state

In the previous exercise, we have shown that the energy-momentum tensor of a perfect fluid in an arbitrary coordinate system is

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}\ , \tag{15}$$

where ρ is the energy density, p the pressure and u^μ the four-velocity of the medium. In an arbitrary curved background, the conservation law $\nabla_\nu T^{\mu\nu} = 0$ implies

$$\partial_\nu T^{\mu\nu} + \Gamma^\mu{}_{\nu\lambda}T^{\lambda\nu} + \Gamma^\nu{}_{\nu\lambda}T^{\mu\lambda} = 0\ . \tag{16}$$

Now we consider the case $\mu = 0$. Using eq. (15) in the rest frame of the fluid and the fact that the FLRW metric fulfills $g_{\mu\nu} = 0$ for $\mu \neq \nu$, we get

$$\begin{aligned} &\partial_\nu T^{0\nu} + \Gamma^0{}_{\nu\lambda}T^{\lambda\nu} + \Gamma^\nu{}_{\nu\lambda}T^{0\lambda} = 0 \\ \Rightarrow\ &\partial_0 T^{00} + \Gamma^0{}_{00}T^{00} + \Gamma^0{}_{ij}T^{ij} + \Gamma^\nu{}_{\nu 0}T^{00} = 0 \\ \Rightarrow\ &\partial_0 T^{00} + 3\Gamma^0{}_{11}T^{11} + 3\Gamma^1{}_{10}T^{00} + 2\Gamma^0{}_{00}T^{00} = 0\ , \end{aligned} \tag{17}$$

where we used isotropy in the last step. Plugging in the Christoffel symbols of the spatially flat FLRW metric, as derived in problem 3 of sheet 1, yields

$$\Rightarrow \frac{d\rho}{dt} + 3RR'\frac{1}{R^2}p + 3\frac{R'}{R}\rho = 0$$
$$\Rightarrow \frac{d}{dt}(R^3\rho) + p\frac{dR^3}{dt} = 0 \ . \tag{18}$$

1. Substituting the dependence $p = w\rho$ into (18) gives
$$\frac{d}{dt}(R^3\rho) + w\rho\frac{dR^3}{dt} = 0 \ . \tag{19}$$
That leads to
$$\frac{1}{(w+1)\rho}\frac{d\rho}{dt} = -\frac{1}{R^3}\frac{d(R^3)}{dt} \ , \tag{20}$$
which has the solution
$$\rho(R) \sim \frac{1}{R^{3(1+w)}} \ . \tag{21}$$
2. Substituting $\rho(R)$ into Friedmann equation,
$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho(R), \tag{22}$$
we have,
$$R(t) \sim t^\alpha, \quad \alpha = \frac{2}{3}\frac{1}{1+w}, \tag{23}$$
and hence
$$\rho(t) \sim \frac{1}{t^2}. \tag{24}$$
Since $\alpha > 0$, $R(t)$ goes to zero as $t \to 0$, while $\rho(t)$ diverges.
3. Differentiating Eq.(23), one finds,
$$\ddot{R}(t) \sim \alpha(\alpha - 1)t^{\alpha-2}. \tag{25}$$
The Universe expands with acceleration if $\alpha - 1 > 0$, or, equivalently, if $w < -1/3$.

3. Einstein Universe

From the lectures we know that in the case of Einstein's universe

$$R_0 = \frac{1}{\sqrt{\lambda}} = \frac{c}{\sqrt{4\pi G\rho}} \ ,$$

where we have restored the speed of light c for numerical estimations. For the cosmological constant, we have

$$\lambda = \frac{1}{R_0^2} = 3.44 \times 10^{-53} \ \frac{1}{\mathrm{m}^2} = 1.33 \times 10^{-84} \ \mathrm{GeV}^2.$$

To obtain this equality, we multiplied by $\hbar^2 c^2$ and transformed J in GeV with $1\,\mu\mathrm{m} = 0.81\,\frac{1}{\mathrm{eV}}$. For the matter density:

$$\rho = \frac{c^2}{4\pi R^2 G} = 3.7 \times 10^{-27}\,\mathrm{kg/m}^3 = 1.59 \times 10^{-47}\,\mathrm{GeV}^4 \ .$$

To get the last equality, we multiplied by $\hbar^3 c^5$ and transformed J in GeV.