RELATIVITY AND COSMOLOGY II

Solutions to Problem Set 1 23th February 2024

1. Homogeneous and isotropic space

We know from the lecture that homogeneous and isotropic space with constant spatial curvature can be described by the metric

$$ds^2 = R^2 \left(\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \; d\phi^2 \right) \right), \tag{1}$$

with $k = 1, 0, -1$. Metrics which describe the same homogeneous spaces with constant spatial curvature should therefore be obtained by a change of variables. Having the term $d\chi^2$ with coefficient 1 requires

$$\frac{dr}{\sqrt{1 - kr^2}} = d\chi. \tag{2}$$

We can solve these equations for the different value of k. We find

$$\begin{aligned} k &= 1 & \rightarrow \quad & r = \sin\chi, \\ k &= 0 & \rightarrow \quad & r = \chi, \\ k &= -1 & \rightarrow \quad & r = \sinh\chi. \end{aligned}$$

With these changes of coordinates the metric takes exactly the forms given in the exercise. Therefore, these describe the same type of homogeneous and isotropic space.

2. Volume in curved spacetime

1. The volume is given by

$$V = \int d^3x \sqrt{\gamma}, \tag{3}$$

with

$$\gamma = \det \begin{pmatrix} \dfrac{1}{1 - \frac{r^2}{R^2}} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} = \frac{r^4 \sin^2\theta}{1 - r^2/R^2}. \tag{4}$$

Therefore, we have

$$\begin{aligned} V &= \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{r^2 \sin\theta}{\sqrt{1 - r^2/R^2}} \\ &= 4\pi \int_0^R dr \frac{r^2}{\sqrt{1 - r^2/R^2}} \\ &= 4\pi R^3 \int_0^{\pi/2} d\chi \sin^2\chi = \pi^2 R^3. \end{aligned} \tag{5}$$

2. Again, the volume is given by

$$V = \int d^3x \sqrt{\gamma}, \tag{6}$$

and this time

$$\gamma = \det \begin{pmatrix} R^2 & 0 & 0 \\ 0 & R^2 \sin^2 \chi & 0 \\ 0 & 0 & R^2 \sin^2 \chi \sin^2 \theta \end{pmatrix} = R^6 \sin^4 \chi \sin^2 \theta. \tag{7}$$

Therefore, we have

$$\begin{aligned} V &= R^3 \int_0^\pi d\chi \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin^2 \chi \sin \theta \\ &= 4\pi R^3 \int_0^\pi d\chi \sin^2 \chi = 2\pi^2 R^3. \end{aligned} \tag{8}$$

3. The difference of the factor of 2 is a result of the fact that the first set of coordinates only covers half of a 3-sphere. Let's begin with definition of 3-sphere in $\mathbb{R}^4$

$$x^2 + y^2 + z^2 + w^2 = R^2. \tag{9}$$

We now rewrite the last three variables into coordinates of a 2-sphere:

$$\begin{aligned} y &= r \sin\theta \sin\varphi, \\ z &= r \sin\theta \cos\varphi, \\ w &= r \cos\theta. \end{aligned} \tag{10}$$

The 3-sphere equation for points on the sphere becomes

$$x^2 + r^2 = R^2, \tag{11}$$

with $r > 0$. This gives two branches of solutions

$$x = \pm\sqrt{R^2 - r^2}, \tag{12}$$

both of which have coordinates range of $r \in [0, R), \theta \in [0, \pi), \phi \in [0, 2\pi)$.

To compute the metric components, one uses chain rule $dx_i = \frac{\partial x_i}{\partial x'_j} dx'_j$ where x'_j are r, θ, ϕ

$$\begin{aligned} dx &= \frac{\pm r dr}{\sqrt{R^2 - r^2}}, \\ dy &= \sin\theta \sin\phi \; dr + r \cos\theta \sin\varphi \; d\theta + r \sin\theta \cos\phi \; d\varphi, \\ dz &= \sin\theta \cos\phi \; dr + r \cos\theta \cos\varphi \; d\theta - r \sin\theta \sin\phi \; d\varphi, \\ dw &= \cos\theta dr - r \sin\theta d\theta, \end{aligned} \tag{13}$$

where the sign in front of the first of the 1-forms depends on the branch of the solution. By substitution it into the Euclidean metric one gets

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2 = \frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2. \tag{14}$$

This is exactly the metric from the first part and now one sees it covers only half of the hypersphere. However, the equation (11) can be solved in another way. The parametrization

$$\begin{aligned} x &= R\cos\chi, \\ r &= R\sin\chi, \end{aligned} \tag{15}$$

with $\chi \in [0, \pi)$ covers the space of both $x \geq 0$ and $x < 0$. This leads to

$$\begin{aligned} dx &= -R\sin\chi\ d\chi, \\ dy &= R\left(\cos\chi\sin\theta\sin\varphi\ d\chi + \sin\chi(\cos\theta\sin\varphi\ d\theta + r\sin\theta\cos\phi\ d\varphi)\right), \\ dz &= R\left(\cos\chi\sin\theta\cos\varphi\ d\chi + \sin\chi(\cos\theta\cos\varphi\ d\theta - r\sin\theta\sin\phi\ d\varphi)\right), \\ dw &= R\left(\cos\chi\cos\theta\ d\chi + \sin\chi(\sin\theta d\theta)\right), \end{aligned} \tag{16}$$

and to the metric

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2 = R^2\Bigg(d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\ d\phi^2\right)\Bigg), \tag{17}$$

which covers the entire hypersphere, not half, like the previous one. That explains the discrepancy in results.

3. Friedmann–Lemaître–Robertson–Walker (FLRW) metric

First we consider the flat space case, with the line element given by

$$ds^2 = -(dx^0)^2 + a^2(x^0)\sum_i(dx^i)^2\ , \tag{18}$$

where for later convenience we introduced the shorthand notation

$$\sum_i(dx^i)^2 = [(dx^1)^2 + (dx^2)^2 + (dx^3)^2]. \tag{19}$$

1. The metric is

$$g_{\mu\nu} = \mathrm{diag}\big[-1, a^2, a^2, a^2\big], \tag{20}$$

so

$$g^{\mu\nu} = \mathrm{diag}\big[-1, a^{-2}, a^{-2}, a^{-2}\big]. \tag{21}$$

2. The action for a classical particle with mass m is

$$S = m\int ds = m\int dp\ \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}, \tag{22}$$

where a dot denotes differentiation with respect to the affine parameter p. Now, to get the equations of motion, we can as well vary the simpler action

$$S = m\int d\tau\ g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \int d\tau\ F(x,\dot{x}), \tag{23}$$

where τ corresponds to proper time. By introducing the explicit form of the metric we find

$$F(x,\dot{x}) = m\Big[-(\dot{x}^0)^2 + a^2\sum_i(\dot{x}^i)^2\Big]. \tag{24}$$

Using the Euler-Lagrange equations

$$\frac{d}{d\tau}\frac{\partial F}{\partial \dot{x}^\mu} = \frac{\partial F}{\partial x^\mu}, \tag{25}$$

we find (use that $\frac{d}{d\tau} = \frac{\partial}{\partial \tau} + \frac{\partial}{\partial x^0}\frac{\partial x^0}{\partial \tau} + \sum_i \frac{\partial}{\partial x^i}\frac{\partial x^i}{\partial \tau}$)

$$\begin{aligned} \ddot{x}^0 &= -aa' \sum_i (\dot{x}^i)^2 \quad \text{for } \mu = 0, \\ \ddot{x}^i &= -2\frac{a'}{a}\dot{x}^0\dot{x}^i \qquad \text{for } \mu = 1,2,3, \end{aligned} \tag{26}$$

where a prime denotes derivative with respect to x^0.

3. The Christoffel symbols are defined as

$$\ddot{x}^\lambda = -\Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu. \tag{27}$$

By identification, the non-zero Γ's are

$$\Gamma^0_{ii} = aa' \qquad \text{and} \qquad \Gamma^i_{0i} = \Gamma^i_{i0} = \frac{a'}{a}. \tag{28}$$

The ${\Gamma^k}_{ij}$, $i, j, k = 1, 2, 3$, are zero because the spatial part of the metric is flat. Let us check the above results with the usual formula

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\kappa\lambda}\left(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}\right).$$

For Γ^i_{0i}, we find

$$\begin{aligned} \Gamma^i_{0i} &= \frac{1}{2}g^{i\kappa}\left(\partial_0 g_{i\kappa} + \partial_i g_{0\kappa} - \partial_\kappa g_{0i}\right) \\ &= \frac{1}{2}g^{ii}\partial_0 g_{ii} \\ &= \frac{1}{2}a^{-2}\partial_0 a^2 = \frac{a'}{a}, \end{aligned}$$

and for ${\Gamma^0}_{ii}$

$$\begin{aligned} \Gamma^0_{ii} &= \frac{1}{2}g^{0\kappa}\left(\partial_i g_{i\kappa} + \partial_i g_{i\kappa} - \partial_\kappa g_{ii}\right) \\ &= -\frac{1}{2}g^{00}\partial_0 g_{ii} \\ &= -\frac{1}{2}\partial_0 a^2 = aa'. \end{aligned}$$

4. For calculating the Ricci tensor $R_{\mu\nu}$, we use the formula

$$R_{\mu\nu} = {R^\rho}_{\mu\rho\nu} = \partial_\rho\Gamma^\rho_{\mu\nu} - \partial_\nu\Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\kappa\rho}\Gamma^\kappa_{\mu\nu} - \Gamma^\rho_{\kappa\nu}\Gamma^\kappa_{\mu\rho}, \tag{29}$$

where ${R^\alpha}_{\mu\rho\nu}$ is the Riemann tensor. We get

$$R_{00} = {R^\kappa}_{0\kappa0} = -3\frac{a''}{a} \qquad \text{and} \qquad R_{ii} = {R^\kappa}_{i\kappa i} = aa'' + 2(a')^2. \tag{30}$$

Note: $R^i_{jij} = 0$ for $i = j$; that's why there is only a factor of 2 in the $(a')^2$ term.

5. Using the above, we see that the scalar curvature is

$$R = g^{\mu\nu} R_{\mu\nu} = 6 \left[\frac{a''}{a} + \left(\frac{a'}{a} \right)^2 \right] \tag{31}$$

6. Finally, the non zero components of the Einstein tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \ , \tag{32}$$

are

$$G_{00} = -3\frac{a''}{a} + 3 \left[\frac{a''}{a} + \left(\frac{a'}{a} \right)^2 \right] = 3 \left(\frac{a'}{a} \right)^2 , \tag{33}$$

$$G_{ii} = aa'' + 2(a')^2 - 3a^2 \left[\frac{a''}{a} + \left(\frac{a'}{a} \right)^2 \right] = -2aa'' - (a')^2. \tag{34}$$

We extracted all the information contained in the metric. The tensor $G_{\mu\nu}$ contains the geometric part of the Einstein equation $G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}$, where $T_{\mu\nu}$ is the energy momentum tensor and Λ is the cosmological constant.

Now we move to the curved space,

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right] . \tag{35}$$

1. The metric is

$$g_{\mu\nu} = \text{diag} \left[-1, \frac{a^2}{1-kr^2}, a^2 r^2, a^2 r^2 \sin^2\theta \right] , \tag{36}$$

so

$$g^{\mu\nu} = \text{diag} \left[-1, \frac{1-kr^2}{a^2}, \frac{1}{a^2 r^2}, \frac{1}{a^2 r^2 \sin^2\theta} \right] . \tag{37}$$

2. The Lagrangian is given by

$$F(x, \dot{x}) = m \left(-\dot{t}^2 + \frac{a^2}{1-kr^2} \dot{r}^2 + a^2 r^2 \dot{\theta}^2 + a^2 r^2 \sin^2\theta \dot{\phi}^2 \right) . \tag{38}$$

Thus, the equations of motion are

$$\begin{aligned} \ddot{t} &= -aa' \left(\frac{\dot{r}^2}{1-kr^2} + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2 \right) , \\ \ddot{r} &= r(1-kr^2) \left[\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2 \right] - k \frac{r\dot{r}^2}{1-kr^2} - 2\frac{a'}{a} \dot{t}\dot{r}, \\ \ddot{\theta} &= \sin\theta \cos\theta \dot{\phi}^2 - 2\frac{\dot{r}}{r} \dot{\theta} - 2\frac{a'}{a} \dot{t}\dot{\theta}, \\ \ddot{\phi} &= -2\frac{\dot{r}}{r} \dot{\phi} - 2\frac{\cos\theta}{\sin\theta} \dot{\theta}\dot{\phi} - 2\frac{a'}{a} \dot{t}\dot{\phi}. \end{aligned} \tag{39}$$

3. The non zero Christoffel symbols are

$$
\begin{aligned}
\Gamma^{t}_{rr} &= \frac{aa'}{1-kr^2}, \\
\Gamma^{t}_{\theta\theta} &= aa'r^2, \\
\Gamma^{t}_{\phi\phi} &= aa'r^2\sin^2\theta, \\
\Gamma^{r}_{rt} &= \frac{a'}{a}, \\
\Gamma^{r}_{rr} &= \frac{kr}{1-kr^2}, \\
\Gamma^{r}_{\theta\theta} &= -r\left(1-kr^2\right), \\
\Gamma^{r}_{\phi\phi} &= -r\left(1-kr^2\right)\sin^2\theta, \\
\Gamma^{\theta}_{\theta t} &= \frac{a'}{a}, \\
\Gamma^{\theta}_{\theta r} &= r^{-1}, \\
\Gamma^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta, \\
\Gamma^{\phi}_{\phi t} &= \frac{a'}{a}, \\
\Gamma^{\phi}_{\phi r} &= r^{-1}, \\
\Gamma^{\phi}_{\phi\theta} &= \frac{\cos\theta}{\sin\theta}.
\end{aligned}
\tag{40}
$$

4. Again using eq. (29), we compute the Ricci tensor

$$
\begin{aligned}
R_{tt} &= -3\frac{a''}{a}, \\
R_{rr} &= \frac{aa''+2k+2(a')^2}{1-kr^2}, \\
R_{\theta\theta} &= r^2\left(aa''+2k+2(a')^2\right), \\
R_{\phi\phi} &= r^2\sin^2\theta\left(aa''+2k+2(a')^2\right).
\end{aligned}
\tag{41}
$$

5. The scalar curvature is

$$
R = g^{\mu\nu}R_{\mu\nu} = 6\left[\frac{a''}{a} + \left(\frac{a'}{a}\right)^2 + \frac{k}{a^2}\right]. \tag{42}
$$

   We remark that the *spatial* curvature modifies the *space-time* curvature by introducing the last term.

6. The Einstein tensor components are

$$
\begin{aligned}
G_{tt} &= 3\left[\left(\frac{a'}{a}\right)^2 + \frac{k}{a^2}\right], \\
G_{rr} &= -\frac{2aa''+(a')^2+k}{1-kr^2}, \\
G_{\theta\theta} &= -r^2\left(2aa''+(a')^2+k\right), \\
G_{\phi\phi} &= -r^2\sin^2\theta\left(2aa''+(a')^2+k\right).
\end{aligned}
\tag{43}
$$