RELATIVITY AND COSMOLOGY II

Solutions to Problem Set 14

1. de Sitter coordinate systems

1. From the flat metric in Minkowski space

$$ds^2 = -dX_0^2 + dX_1^2 + \dots dX_d^2 \tag{1}$$

we obtain

$$ds^2 = -(\cosh t)^2 dt^2 + \sum_{i=1}^{d} \left((\cosh t)^2 (d\omega^i)^2 + \cosh t \sinh t dt (\omega^i d\omega^i) + (\omega^i)^2 (\sinh t)^2 dt^2 \right) \tag{2}$$

Now we use the following identities

$$\sum_{i=1}^{d} (\omega^i)^2 = 1,$$
$$\sum_{i=1}^{d} (d\omega^i)^2 = d\Omega_{d-1}^2,$$
$$\sum_{i=1}^{d} \omega^i d\omega^i = 0$$

to obtain the final expression for the metric

$$ds^2 = -dt^2 + (\cosh t)^2 d\Omega_{d-1}^2 \tag{3}$$

In these coordinates dS_d looks like a $(d-1)$–dimensional sphere which starts out infinitely large at $t = -\infty$, then shrinks to the minimal finite size at $t = 0$, to then grow again to infinite size as $t \to +\infty$.

2. By the same procedure we obtain that the metric in these coordinates looks like

$$ds^2 = -dt^2 + e^{-2t} (dx^i)^2 \tag{4}$$

These coordinates cover only a half of the dS_d space. The surfaces of constant t are infinite $d-1$-dimensional planes with flat metric.

3. For the static patch coordinates we obtain

$$ds^2 = -(1-r)^2 dt^2 + \frac{dr^2}{1-r^2} + r^2 d\Omega_{d-1}^2 \tag{5}$$

Note that here ω^i parametrize $(d-2)$-dimensional sphere. In these coordinate system $\frac{\partial}{\partial t}$ is a Killing vector. Also in these coordinates one can explicitly see the appearance of an event horizons at $r^2 = 1$.

2. Scalar field in FLRW spacetime

1. We begin from the action for a minimally coupled scalar field (in the $(-+++)$ convention for the metric that we use in this course):

$$-\int d^4x\sqrt{-g}\left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)+V(\phi)\right]. \tag{6}$$

The energy-momentum tensor is given as

$$T_{\mu\nu}\equiv-\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{7}$$

From GRI exercises we know that

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}=-\frac{1}{2}\sqrt{-g}\;g_{\mu\nu}, \tag{8}$$

so that we get

$$\begin{aligned}
T_{\mu\nu}&=\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left[\sqrt{-g}\left(\frac{1}{2}\left(\partial_\alpha\phi\right)g^{\alpha\beta}\left(\partial_\beta\phi\right)+V(\phi)\right)\right]\\
&=\frac{2}{\sqrt{-g}}\Bigg[\left(\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right)\left(\frac{1}{2}\left(\partial_\alpha\phi\right)g^{\alpha\beta}\left(\partial_\beta\phi\right)+V(\phi)\right)\\
&+\sqrt{-g}\frac{\delta}{\delta g^{\mu\nu}}\left(\frac{1}{2}\left(\partial_\alpha\phi\right)g^{\alpha\beta}\left(\partial_\beta\phi\right)+V(\phi)\right)\Bigg]\\
&=-\frac{2}{\sqrt{-g}}\left[\frac{1}{2}\sqrt{-g}g_{\mu\nu}\left(\frac{1}{2}\left(\partial_\alpha\phi\right)g^{\alpha\beta}\left(\partial_\beta\phi\right)+V(\phi)\right)+\sqrt{-g}\cdot\frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi)\right]\\
&=-g_{\mu\nu}\left[\frac{1}{2}\left(\partial_\alpha\phi\right)g^{\alpha\beta}\left(\partial_\beta\phi\right)+V(\phi)\right]+(\partial_\mu\phi)(\partial_\nu\phi).
\end{aligned} \tag{9}$$

Inserting the FLRW metric $g_{\mu\nu}=\mathrm{diag}\,[-1,a^2,a^2,a^2]$ (as well as its inverse $g^{\mu\nu}=\mathrm{diag}\,[-1,a^{-2},a^{-2},a^{-2}]$) and using spatial homogeneity of the field ϕ, we get for the diagonal elements of the energy-momentum tensor

$$T_{00}=-1\left[\frac{1}{2}\dot{\phi}^2-V(\phi)\right]+\dot{\phi}^2=\frac{1}{2}\dot{\phi}^2+V(\phi), \tag{10}$$

and

$$T_{ii}=a^2\left[\frac{1}{2}\dot{\phi}^2-V(\phi)\right], \tag{11}$$

where no sum over i (and j) is implied here. The off-diagonal elements vanish:

$$T_{0i}=T_{ij}|_{i\neq j}=0. \tag{12}$$

Finally, we raise all indices. This is straightforward since the metric is diagonal:

$$T^{00}=g^{0\alpha}g^{0\beta}T_{\alpha\beta}=(g^{00})^2T_{00}=\frac{1}{2}\dot{\phi}^2+V(\phi), \tag{13}$$

$$T^{ii}=g^{i\alpha}g^{i\beta}T_{\alpha\beta}=(g^{ii})^2T_{ii}=a^{-2}\left[\frac{1}{2}\dot{\phi}^2-V(\phi)\right], \tag{14}$$

$$T^{0i}=T^{ij}|_{i\neq j}=0, \tag{15}$$

where sum over α and β is implied whereas we do not sum over i and j.

The energy-momentum tensor of a perfect fluid in its rest frame is (see sheet 3, exercise 1):

$$T^{\mu\nu} = (\rho + p)\delta_0^\mu \delta_0^\nu + p g^{\mu\nu}. \tag{16}$$

This implies

$$\rho = T^{00} = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{17}$$

and

$$p = \frac{a^2}{3}\left(T^{11} + T^{22} + T^{33}\right) = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{18}$$

Finally, we can compute the state parameter:

$$w = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{19}$$

2. The Friedmann equation gives

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{20}$$

For accelerated expansion, we therefore want $\rho + 3p < 0$, which implies

$$\dot{\phi}^2 < V(\phi). \tag{21}$$

3. For the equation of motion, we insert Eqs. (17) and (18) to the continuity equation:

$$\dot{\rho} + 3H(\rho + p) = 0 \qquad \Rightarrow \qquad \dot{\phi}\ddot{\phi} + \frac{dV}{d\phi}\dot{\phi} + 3H \cdot \dot{\phi}^2 = 0$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \tag{22}$$

Indeed, this is the equation of motion for the scalar field. We could have found it directly from varying the first action in Eq. (6) with respect to ϕ. Finally, the Friedmann equation implies

$$H^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \tag{23}$$

3. Behavior of the inflaton

The equations of motion for a free massive scalar field are (see previous exercise)

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0\ , \tag{24}$$

$$H^2 = \frac{8\pi G}{6}\left(\dot{\phi}^2 + m^2\phi^2\right)\ . \tag{25}$$

1. Solving the second equation for H and putting it into the first, we get

$$\ddot{\phi} + \sqrt{12\pi G}\sqrt{\dot{\phi}^2 + m^2\phi^2}\dot{\phi} + m^2\phi = 0 \ . \tag{26}$$

   This is a nonlinear second order differential equation with no explicit time dependence. It can therefore be reduced to a first order differential equation for $\dot{\phi}(\phi)$. Using $\ddot{\phi} = \dot{\phi}\frac{d\dot{\phi}}{d\phi}$, we get

$$\frac{d\dot{\phi}}{d\phi} = -\frac{\sqrt{12\pi G}\sqrt{\dot{\phi}^2 + m^2\phi^2}\dot{\phi} + m^2\phi}{\dot{\phi}} \ . \tag{27}$$

2. (a) <u>"Ultra-hard" period ($\dot{\phi} \gg m\phi$ and $\dot{\phi}^2 \gg \frac{m^2}{\sqrt{G}}\phi$)</u>

   This is the situation where the potential energy is small compared to the kinetic energy. In this approximation, Eq. (27) becomes

$$\frac{d\dot{\phi}}{d\phi} \approx -\sqrt{12\pi G}\dot{\phi} \ , \tag{28}$$

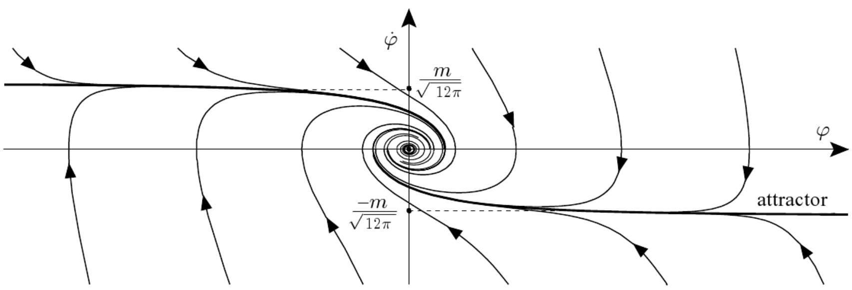
   so the solution is damped exponentially

$$\dot{\phi} \approx C \exp(-\sqrt{12\pi G}\phi) \ , \tag{29}$$

   where C a constant of integration. We can immediately solve this relation for $\phi(t)$ to obtain

$$\phi(t) = \frac{1}{\sqrt{12\pi G}} \log\left[C\sqrt{12\pi G}t\right] \ . \tag{30}$$

   The above relations tell us that even if $\dot{\phi}$ had a large initial value, it decays exponentially faster than the value of the scalar field itself. Therefore, the attractor is reached very quickly and this enlarges the set of initial conditions which lead to an inflationary stage.



   In order to find the Hubble parameter, we use Eq. (25), which in the "ultra-hard" limit reads

$$H^2 \approx \frac{4\pi G}{3}\dot{\phi}^2 \ . \tag{31}$$

   Plugging (30) into the above, we obtain

$$H^2 \approx \frac{1}{9t^2} \quad \rightarrow \quad H \approx \frac{1}{3t} \ . \tag{32}$$

(b) "Slow-roll" period ($d\dot{\phi}/d\phi \approx 0$ and $\dot{\phi}^2 \ll m^2\phi^2$)

In this limit, Eq. (27) gives

$$-\sqrt{12\pi G} - \frac{m}{\dot{\phi}} \approx 0 \quad \rightarrow \quad \dot{\phi} \approx -\frac{m}{\sqrt{12\pi G}} \ . \tag{33}$$

Using this result, we find that

$$\phi(t) \approx \phi_0 - \frac{mt}{\sqrt{12\pi G}} \ , \tag{34}$$

where ϕ_0 is some initial value. Correspondingly, the Hubble parameter during the "slow-roll" period is found from (25) to be equal to

$$H \approx H_0 - \frac{m^2 t}{3} \ , \tag{35}$$

where we defined $H_0 = \sqrt{\frac{4\pi G}{3}} m\phi_0$. A useful check at this point is to compute the scale factor a. We find

$$H \equiv \frac{\dot{a}}{a} \approx H_0 - \frac{m^2 t}{3} \quad \rightarrow \quad a \propto e^{H_0 t - \frac{m^2 t^2}{6}} \ . \tag{36}$$

For $t \ll (\sqrt{G}\phi_0)^2/H_0$, we can neglect the second term in the exponent and obtain exponential expansion of the universe, as we should. (Even for larger values of t, exponential expansion will continue approximately albeit with a changing value of H.)

3. We are now asked to compute the Hubble parameter when the potential and kinetic terms are of the same order of magnitude. To do so, it is more convenient to work with the original system of equation for the inflaton.

   From eq. (25), we see that

$$\dot{\phi}^2 + m^2\phi^2 = \frac{3H^2}{4\pi G} \ . \tag{37}$$

   We notice that the above equation can be solve using the following change of variables:

$$\dot{\phi} = \sqrt{\frac{3}{4\pi G}} H \sin\theta \ , \tag{38}$$

$$\phi = \frac{1}{m}\sqrt{\frac{3}{4\pi G}} H \cos\theta \ . \tag{39}$$

   Combining these two equations, we find

$$\dot{H}\cos\theta - H\dot{\theta}\sin\theta = mH\sin\theta \ . \tag{40}$$

   We now turn to eq. (24)

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0 \ . \tag{41}$$

   Using

$$\ddot{\phi} = \frac{1}{2\dot{\phi}}\frac{d}{dt}\dot{\phi}^2 \quad \text{and} \quad \phi = \frac{1}{2\dot{\phi}}\frac{d}{dt}\phi^2 \ , \tag{42}$$

the above becomes

$$\frac{d}{dt}\dot{\phi}^2 + 6H\dot{\phi}^2 + m^2\frac{d}{dt}\phi^2 = 0 \ . \tag{43}$$

Plugging in the expressions for $\dot{\phi}$ and ϕ from (38) and (39), we find

$$\frac{d}{dt}\left(H^2\right) = -6H^3\sin^2\theta \qquad \Rightarrow \qquad \dot{H} = -3H^2\sin^2\theta \ . \tag{44}$$

Finally, we replace $\dot{H}$ in (40) to obtain

$$\dot{\theta} \approx -m \ , \tag{45}$$

where we averaged over timescales that are much bigger than m^{-1} and simultaneously much smaller than H^{-1}. This is possible since $H \ll m$ and it allows us to drop the oscillatory term. We arrive at:

$$\theta \approx -mt \ , \tag{46}$$

therefore

$$\dot{H} = -3H^2\sin^2(mt) \ . \tag{47}$$

Finally, we integrate this equation and get

$$\frac{1}{3H} = \frac{t}{2}\left(1 - \frac{\sin(2mt)}{2mt}\right) \approx \frac{t}{2} \ , \tag{48}$$

for $mt \gg 1$. Therefore

$$H \approx \frac{2}{3t} \ . \tag{49}$$

This expression suggests that we are inside a matter dominated period.

4. We have seen that the temperature T is related to the energy density ρ as

$$\rho = \frac{\pi^2}{30}g_*(T)T^4 \ ,$$

where $g_*(T)$ is the number of relativistic degrees of freedom at a given temperature.

We are asked to estimate the temperature at the end of inflation, i.e. when the slow roll condition is saturated:

$$\dot{\phi}_{end}^2 \approx m^2\phi_{end}^2 \ , \tag{50}$$

where we denoted ϕ_{end} the value of the field at the end of inflation. This means that the energy density will be

$$\rho(\phi_{end}) = \frac{1}{2}\left(\dot{\phi}_{end}^2 + m^2\phi_{end}^2\right) \approx m^2\phi_{end}^2 \ . \tag{51}$$

Since we want all the energy to be transferred to the SM particles, $g_*(T_{end}) \sim 100$, therefore

$$T_{end} \approx 0.4\sqrt{m\phi_{end}} \ . \tag{52}$$

In part 2 we saw that during slow-roll, $|\dot{\phi}| \approx \frac{m}{\sqrt{12\pi G}}$, so from (50) we find that

$$\phi_{end} \approx \frac{1}{\sqrt{12\pi G}} \ . \tag{53}$$

This leads to

$$T_{end} \approx 0.2\sqrt{\frac{m}{\sqrt{G}}} = 0.2\sqrt{mM_{\rm Pl}} \ . \tag{54}$$