RELATIVITY AND COSMOLOGY II

Solutions to Problem Set 13

1. Cold Dark Matter perturbations in radiation domination

1. First of all, we need to Fourier transform and write the equations in momentum space:

$$\delta\rho_\lambda' + 3\frac{a'}{a}\left(\delta\rho_\lambda + \delta p_\lambda\right) - \left(\rho_\lambda + p_\lambda\right)\left(k^2 v_\lambda + 3\Phi'\right) = 0, \tag{1}$$

$$\left[\left(\rho_\lambda + p_\lambda\right) v_\lambda\right]' + 4\frac{a'}{a}\left(\rho_\lambda + p_\lambda\right) v_\lambda + \delta p_\lambda + \left(\rho_\lambda + p_\lambda\right)\Phi = 0. \tag{2}$$

Secondly, we recall the dispersion relations $\delta p_\lambda = u_s^2(\lambda)\delta\rho_\lambda$ and $p_\lambda = w(\lambda)\rho_\lambda$, where the first relates single-component density perturbations, while the second the background densities. As $w^{CDM} = u_s^{CDM} = 0$ we can neglect immediately all the pressure terms in the equations (1) and (2), and be left with

$$\delta_{CDM}' + \frac{\rho_{CDM}'}{\rho_{CDM}}\delta_{CDM} + 3\frac{a'}{a}\delta_{CDM} - k^2 v_{CDM} - 3\Phi' = 0, \tag{3}$$

$$\frac{\rho_{CDM}'}{\rho_{CDM}} v_{CDM} + v_{CDM}' + 4\frac{a'}{a} v_{CDM} + \Phi = 0. \tag{4}$$

The hint helps us remember that $\frac{\rho_{CDM}'}{\rho_{CDM}} = -3\frac{a'}{a}$, so we are lead to:

$$\delta_{CDM}' - k^2 v_{CDM} = 3\Phi' \tag{5}$$

$$v_{CDM}' + \frac{a'}{a} v_{CDM} = -\Phi. \tag{6}$$

In a radiation dominated universe $a \propto \eta$, so $\frac{a'}{a} = \frac{1}{\eta}$ which finally brings us to the set of equations

$$\delta_{CDM}' - k^2 v_{CDM} = 3\Phi', \tag{7}$$

$$v_{CDM}' + \frac{1}{\eta} v_{CDM} = -\Phi. \tag{8}$$

2. In the homogeneous case, $\Phi = 0$, the equations become

$$\delta_{CDM}' - k^2 v_{CDM} = 0 \tag{9}$$

$$v_{CDM}' + \frac{1}{\eta} v_{CDM} = 0. \tag{10}$$

Equations (10) can be immediately integrated to give

$$v_{CDM} = \frac{A}{k^2\eta};$$

Substituting this back into (9) we have the general solution for δ_{CDM}

$$\delta_{CDM} = A\log(k\eta) + B.$$

In general then, Cold Dark Matter perturbations in a radiation dominated background are expected to grow logarithmically. Regularity conditions at early times however, would impose $A = 0$, leaving the perturbations as constant (another way of phrasing the same concept, is that the logarithmic mode decays very very quickly at early times and so should be neglected).

3. Multiplying both sides of equation (6) by η we can bring it in the form:

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta \cdot v_{CDM}\right) = -\eta \cdot \Phi$$

whose unique solution with the property of being finite for $\eta \to 0$ is

$$v_{CDM} = -\frac{1}{\eta}\int_0^\eta d\tilde{\eta}\tilde{\eta} \cdot \Phi.$$

In particular, $v_{CDM} \to 0$ as $\eta \to 0$.

4. We can plug the solution for v_{CDM} into equation (5), and get

$$\delta'_{CDM} + \frac{k^2}{\eta}\int_0^\eta d\tilde{\eta}\tilde{\eta} \cdot \Phi = 3\Phi'$$

whose solution is formally

$$\delta_{CDM} = 3\Phi(\eta) + \mathbf{C} - k^2 \int_0^\eta \frac{d\eta_1}{\eta_1}\int_0^{\eta_1} d\eta_2\eta_2\Phi(\eta_2),$$

and $\mathbf{C}$ an integration constant. The constant is fixed by the values of δ_{CDM} and Φ for $\eta \to 0$, $\delta_{CDM,(i)}$ and $\Phi_{(i)}$; moreover we can swap the order of integration so that the η_1–integration can be performed, and obtain

$$\delta_{CDM} = \delta_{CDM,(i)} + 3\left(\Phi(\eta) - \Phi_{(i)}\right) - k^2 \int_0^\eta d\eta_2\eta_2\Phi(\eta_2)\log\left(\frac{\eta}{\eta_2}\right).$$

The last integral of the above expression contains two modes, as $\log\left(\frac{\eta}{\eta_2}\right) = \log\eta - \log\eta_2$. The leading behaviour at early times will be given by the first term, which presents a divergence for $\eta \to 0$, while the second amounts to a constant.

In particular then, we need to solve the integral

$$\log\eta \cdot k^2 \int_0^\eta d\eta_2\eta_2\Phi(\eta_2),$$

which, by a reasonable change of integration variables becomes

$$-\frac{3\Phi_{(i)}}{(u_s^{rad})^2}\log(u_s^{rad}k\eta) \cdot \int_0^{u_s^{rad}k\eta} dx\left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right).$$

For $x \gg 1$ the potential rapidly decreases (in point 2 we even approximated it to be 0), so we can extend the integration limit to $+\infty$, and solve the definite integral

$$\int_0^{+\infty} dx \left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right) = \int_0^{+\infty} dx \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{x}\right) = -1.$$

Moreover, $u_s^{rad} = \frac{1}{\sqrt{3}}$, so the required leading behavior is

$$\delta_{CDM} = \delta_{CDM,(i)} - 9\Phi_{(i)} \cdot \log(u_s^{rad} k\eta),$$

which shows a logarithmic pattern, as famously expected.

2. Sachs-Wolfe effect

1. In a inhomogeneous universe, as the photon travels through it will gain or lose energy. Its free evolution is governed by the geodesic equation

$$P^\mu := \frac{dx^\mu}{d\lambda} \qquad \frac{dP^\mu}{d\lambda} = -\Gamma^\mu_{\nu\rho} P^\nu P^\rho,$$

   We will adopt the coordinate system in which the metric takes the conformal Newtonian form

$$ds^2 = a^2(\eta)\left[-(1-2\Phi)d\eta^2 + (1-2\Psi)\delta_{ij}dx^i dx^j\right],$$

   so in particular, the 0-component of the geodesic equation will take the form

$$\begin{aligned}
\frac{dP^0}{d\eta} &= \frac{1}{P^0}\frac{dP^0}{d\lambda} = -\Gamma^0_{\nu\rho}\frac{P^\nu P^\rho}{P^0} \\
&= -\Gamma^0_{00}P^0 + 2\Gamma^0_{0i}P^i - \Gamma^0_{ij}\frac{P^iP^j}{P^0} \\
&= -\left(\frac{\dot{a}}{a} - \Phi'\right)P^0 + 2\partial_i\Phi P^i - \left[\frac{\dot{a}}{a} - \Psi' - 2\frac{\dot{a}}{a}(\Psi - \Phi)\right]\delta_{ij}\frac{P^iP^j}{P^0}.
\end{aligned}$$

2. In a perturbed universe, the coordinate-momentum P^μ defined above, and the energy-momentum measured by a local observer in their own local frame $\tilde{P}^\mu = (E, \tilde{P}^i)$ are different. We can relate the two the common square invariant

$$\eta_{\mu\nu}\tilde{P}^\mu\tilde{P}^\nu = g_{\mu\nu}P^\mu P^\nu = 0 \implies \begin{cases} P^0 = \frac{E}{\sqrt{-g_{00}}} = \frac{E}{a}(1+\Phi) \\ P^i = \frac{E}{\sqrt{-g_{ii}}}\hat{n}^i = \frac{E}{a}(1+\Psi)\hat{n}^i \end{cases}$$

   Substituting back into the geodesic equation we find the evolution for the energy measured by a local observer:

$$\begin{aligned}
\frac{d}{d\eta}\left(\frac{E}{a}(1+\Phi)\right) = &-\left(\frac{\dot{a}}{a} - \Phi'\right)\frac{E}{a}(1+\Phi) + 2\frac{E}{a}(1+\Psi)(\hat{n}\cdot\partial)\Phi \\
&- \left[\frac{\dot{a}}{a} - \Psi' - 2\frac{\dot{a}}{a}(\Psi - \Phi)\right]\frac{E}{a}\cdot\frac{(1+\Psi)^2}{(1+\Phi)^2},
\end{aligned}$$

which at first order reduces to

$$\frac{1}{E}\frac{dE}{d\eta} = -\frac{\dot{a}}{a} + \Psi' + (\hat{n}\cdot\partial)\Phi. \tag{11}$$

Each of the three terms in equation (11) has a clear physical interpretation: the first term gives the redshifting of the photon given by the expansion of the universe, $E \propto a^{-1}$; the second term captures the effect given by the local deformation of the scale factor due to the inhomogeneities $a(\eta) \mapsto \tilde{a}(\vec{x},\eta) = (1+\psi)a(\eta)$, so now the energy of the photon is redshifted as $E \propto \tilde{a}^{-1}$. Finally, the last term describes the gravitational redshift or blueshift as the photon travels out (or respectively, falls in) a gravitational potential well.

3. Using the relation $(\hat{n}\cdot\partial)\Phi = \frac{d\Phi}{d\eta} - \frac{\partial\Phi}{\partial\eta}$ we can rewrite the geodesic equation

$$\frac{d\ln(aE)}{d\eta} = \frac{d\Phi}{d\eta} + \Phi' - \Psi',$$

which can be integrated between our time and the time of recombination, to become

$$\ln(aE)_0 = \ln(aE)_{rec} + \Phi_0 - \Phi_{rec} + \int_{\eta_{rec}}^{\eta_0} d\eta\,(\Phi' - \Psi')\,. \tag{12}$$

We can set Φ_0, as constant shifts in the potential don't change the equations of motion.

After decoupling, the photon distribution maintains the same shape as they simply stream along their geodesics, and since the Bose-Einstein distribution only depends on $\frac{E}{T}$, it means the effective photon temperature goes like $T \propto E$. Therefore we can relate the temperature anisotropies to the energy as

$$aE \propto a\bar{T}\left(1 + \frac{\delta T}{\bar{T}}\right)$$

and expanding the log in (12), we get

$$\left.\frac{\delta T}{T}\right|_0 = \left.\frac{\delta T}{T}\right|_{rec} - \Phi_{rec} + \int_{\eta_{rec}}^{\eta_0} d\eta\,(\Phi' - \Psi')\,. \tag{13}$$

4. Let's examine the contribution to $\left.\frac{\delta T}{T}\right|_{rec}$; there will be fluctuations induced by energy-density fluctuations in the primordial plasma, and by doppler effects due to the motion of the electrons. Neglecting this second contribution, we have that, since $\rho_\gamma \propto T^4$

$$\left.\frac{\delta T}{T}\right|_{rec} = \frac{1}{4}\frac{\delta\rho_\gamma}{\rho_\gamma}.$$

Substituting back in, and taking the working hypothesis of a perfect fluid, we have

$$\left.\frac{\delta T}{T}\right|_0 = \frac{1}{4}\frac{\delta\rho_\gamma}{\rho_\gamma} - \Phi_{rec}.$$

This is called the Sachs-Wolfe effect: it combines the intrinsic temperature fluctuations associated with the density fluctuations on the last scattering surface, with the gravitational redshift undergone by the photon travelling towards us. The neglected integral $\int_{\eta_{rec}}^{\eta_0} d\eta\,(\Phi' - \Psi')$ turns out to be a subdominant contribution, and it is known under the name *integrated* Sachs-Wolfe effect.

Since the decoupling happens in the matter domination era, we can relate Φ to the density fluctuations via the simple relation

$$2\Phi \simeq \frac{\delta\rho}{\rho} \simeq \frac{\delta\rho_m}{\rho_m} = \frac{3}{4}\frac{\delta\rho_\gamma}{\rho_\gamma},$$

where the first equality is given by Einstein equation, the second to the fact that in the matter-domination period, the dominant energy-density contribution is given by the non-relativistic matter component, while the last follows from the adiabatic evolution of density perturbations (which is always the case in a barotropic, perfect fluid).

Putting it all together, this brings to the famous relation

$$\left.\frac{\delta T}{T}\right|_0 = \frac{2}{3}\Phi_{rec} - \Phi_{rec} = -\frac{1}{3}\Phi_{rec}.$$

An alternative way to solve this very last part of the exercise, is to work in the gauge where recombination happens instantaneously at some fixed *energy density*, rather than at some fixed value of the redshift z. In such case, the intrinsic temperature fluctuations at recombinations are due to the fact that recombination doesn't it all happen at the same z, and therefore, recalling $\frac{a}{a_0} := \frac{1}{1+z}$

$$\left.\frac{\delta z}{1+z}\right|_{rec} = -\left.\frac{\delta a}{a}\right|_{rec} = -\left.\frac{\dot{a}}{a}\delta t\right|_{rec} = -\frac{2}{3}\frac{\delta t}{t_{rec}} = -\frac{2}{3}\cdot\delta\left(\frac{1}{1+\Phi_{rec}}\right) = \frac{2}{3}\Phi_{rec}.$$

3. Horizon problem

1. Without inflation, the particle horizon (the distance over which we expect causal contact) at the time of recombination (CMB decoupling) by now makes an angle of approximate 1.5° on the sky. How does that answer change for a universe which is always dominated by radiation?

   Recombination happens at $t_{\rm rec} \approx 3.7 \times 10^5$ years. Now, if there is only a radiation stage in the history of the universe, the scale factor changes in time as

   $$a = a_0\left(\frac{t}{t_0}\right)^{1/2} \propto t^{1/2}. \tag{14}$$

   Then, we have for the horizon at the time of recombination

   $$d_h(t_{\rm rec}) = a(t_{\rm rec})\int_0^{t_{\rm rec}} \frac{c\,\mathrm{d}t}{a(t)} = c\cdot t_{\rm rec}^{1/2}\int_0^{t_{\rm rec}} t^{-1/2}\,dt = 2c\cdot t_{\rm rec}. \tag{15}$$

   The conformal size of this horizon is

   $$\tilde{d}_h(t_{\rm rec}) = \frac{d_h(t_{\rm rec})}{a(t_{\rm rec})} = \frac{2c\cdot t_{\rm rec}}{a(t_{\rm rec})} = \frac{2c}{a_0}(t_0 t_{\rm rec})^{1/2}. \tag{16}$$

The conformal distance from which the relic photons fly towards us since the time of recombination is

$$\tilde{l} = \int_{t_{\rm rec}}^{t_0} \frac{c\,\mathrm{d}t}{a(t)} = \frac{ct_0^{1/2}}{a_0} \int_{t_{\rm rec}}^{t_0} t^{-1/2}\,\mathrm{d}t = \frac{2ct_0}{a_0}\left[1 - \left(\frac{t_{\rm rec}}{t_0}\right)^{1/2}\right] \tag{17}$$

Therefore, the angle θ_h that the decoupling horizon makes on today's sky

$$\theta_h = \frac{\tilde{d}_h(t_{\rm rec})}{\tilde{l}} = \frac{\frac{2c}{a_0}(t_0 t_{\rm rec})^{1/2}}{\frac{2ct_0}{a_0}\left[1 - \left(\frac{t_{\rm rec}}{t_0}\right)^{1/2}\right]} = \left[\left(\frac{t_0}{t_{\rm rec}}\right)^{1/2} - 1\right]^{-1} \approx 5.2 \times 10^{-3}\,\mathrm{rad} \approx 0.3°, \tag{18}$$

where we have used the current age of the universe $t_0 \approx 13.7 \times 10^9$ years.

2. If inflation solves the horizon problem, we want that the particle horizon generated at the end of inflation, which by now has a size of

$$d_h(t_e) \cdot \frac{a_0}{a_e}, \tag{19}$$

is equal or greater than the distance we can look back today

$$l(t_0) = a_0 \int_{t_e}^{t_0} \frac{c\,\mathrm{d}t}{a(t)} = \|a(t) \propto t^{2/3}\| \approx 3ct_0 = \frac{2c}{H_0}. \tag{20}$$

Here we assumed that the dominant contribution to l is made by the matter-dominated epoch. Note that we can receive signals only from some particles which were created after the end of inflation. Therefore the lower integration boundary in Eq. (20) is t_e and not t_i (the initial moment of inflation).

So let us say

$$\frac{d_h(t_e)}{a_e} \simeq \frac{l(t_0)}{a_0}. \tag{21}$$

During inflation, the Hubble parameter $H_{\rm inf}$ is almost constant which leads to the quasiexponential expansion of the universe,

$$a(t) = a_i \exp(H_{\rm inf} t), \tag{22}$$

where again index i denotes the beginning of inflation (we can choose $t_i = 0$). Then, the horizon size at the end of inflation,

$$d_h(t_e) = a_e \int_0^{t_e} \frac{c\,\mathrm{d}t}{a(t)} = \frac{ca_e}{a_i} \int_0^{t_e} e^{-H_{\rm inf} t}\mathrm{d}t = \frac{c}{H_{\rm inf}}\left(\frac{a_e}{a_i} - 1\right) \approx \frac{c}{H_{\rm inf}} \frac{a_e}{a_i}. \tag{23}$$

So let us try to write the equality of Eq. (21) in terms of the number N of e-folds of inflation, $N \equiv \log\left(\frac{a_e}{a_i}\right)$:

$$\begin{aligned}
\frac{c}{H_{\rm inf}}\frac{1}{a_i} &\simeq \frac{2c}{H_0 a_0} \\
\frac{a_e}{a_i} &\simeq 2\frac{H_{\rm inf}}{H_0}\frac{a_e}{a_0} \\
e^N &\simeq 2\frac{H_{\rm inf}}{H_0}\frac{T_0}{T_e}\left(\frac{g_0}{g_e}\right)^{1/3} \\
N &\simeq \log\left(2\frac{H_{\rm inf}}{H_0}\frac{T_0}{T_e}\left(\frac{g_0}{g_e}\right)^{1/3}\right).
\end{aligned} \tag{24}$$

Here we used the entropy conservation law in order to relate the scale factors at the end of inflation and today in terms of corresponding temperatures; g_e and g_0 are the corresponding numbers of relativistic degrees of freedom.

Further, we use the Friedmann equation to find

$$H_{\rm inf} = \sqrt{\frac{8\pi G}{3}\rho} = \|\rho = v^4\| = \sqrt{\frac{8\pi}{3}}\frac{v^2}{M_{\rm Pl}}, \tag{25}$$

where $M_{\rm Pl} = G^{-1/2}$ is the Planck mass. Also, we know that $T_e = v$. Finally, we have the following expression for the number of e-foldings during inflation:

$$N = \log\left[4\sqrt{\frac{2\pi}{3}}\left(\frac{g_0}{g_e}\right)^{1/3}\frac{v}{M_{\rm Pl}}\frac{T_0}{H_0}\right]. \tag{26}$$

3. Now let's try to put in the numbers. The most straightforward way is to express everything in the natural units.

   We have

$$T_0 = 2.73\,{\rm K} \approx 2.3\times10^{-13}\,{\rm GeV}, \tag{27}$$

$$H_0 = 67.7\,{\rm km/(s\cdot Mpc)} \approx 2.2\times10^{-18}\,{\rm s}^{-1} \approx 1.5\times10^{-42}\,{\rm GeV}, \tag{28}$$

$$M_{\rm Pl} = 1.22\times10^{19}\,{\rm GeV}. \tag{29}$$

   We know that

$$g_0 = 2 + \frac{7}{8}\times6\times\frac{4}{11} \approx 3.9. \tag{30}$$

   We will estimate the number of d.o.f. at the end of inflation as the full number of d.o.f. in the Standard Model, i.e.,

$$g_e = 106.75. \tag{31}$$

   Then, we finally get

$$N \approx 24 + \log v[\,{\rm GeV}\,]. \tag{32}$$

   This gives

$$\begin{aligned} v = 10^8\,{\rm GeV} \quad &\Rightarrow \quad N \approx 42,\\ v = 10^{16}\,{\rm GeV} \quad &\Rightarrow \quad N \approx 61. \end{aligned} \tag{33}$$

   This rough estimate comes pretty close to what we derived in the lecture!