RELATIVITY AND COSMOLOGY II

Solutions to Problem Set 12 17th May 2024

1. Linearised Einstein equations for scalar perturbation

In order to derive the general expression for the linearised Einstein tensor we will exploit the fact that the metric $g_{\mu\nu} = a^2 (\eta_{\mu\nu} + h_{\mu\nu})$ and $\gamma_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ are related by a conformal transformation of factor $\phi = \log a$. Henceforth, the Einstein tensor associated to $g_{\mu\nu}$ is related to the one generated by $\gamma_{\mu\nu}$ as

$$G_{\mu\nu}(g) = G_{\mu\nu}(\gamma) - 2\frac{\nabla_\mu \nabla_\nu a}{a} + 4\frac{\partial_\mu a \partial_\nu a}{a^2} + \gamma_{\mu\nu}\gamma^{\lambda\rho}\left(2\frac{\nabla_\lambda \nabla_\rho a}{a} - \frac{\partial_\lambda a \partial_\rho a}{a^2}\right),$$

where the covariant derivatives are evaluated with the metric $\gamma_{\mu\nu}$, but indices need to be raised with $g^{\mu\nu} = \frac{1}{a^2}\gamma^{\mu\nu} = \frac{1}{a^2}(\eta^{\mu\nu} - h^{\mu\nu})$:

$$G^\mu_\nu(g) = \frac{1}{a^2}\left[G^\mu_\nu(\gamma) - 2\gamma^{\mu\lambda}\frac{\nabla_\lambda \nabla_\nu a}{a} + 4\gamma^{\mu\lambda}\frac{\partial_\lambda a \partial_\nu a}{a^2} + \delta^\mu_\nu \gamma^{\lambda\rho}\left(2\frac{\nabla_\lambda \nabla_\rho a}{a} - \frac{\partial_\lambda a \partial_\rho a}{a^2}\right)\right].$$

Linearising this last expression, you can obtain

$$\begin{aligned}\delta G^\mu_\nu =& \frac{1}{a^2}\left[R^\mu_\nu(\gamma) + 2h^{\mu\lambda}\frac{\partial_\lambda \partial_\nu a}{a} - 4h^{\mu\lambda}\frac{\partial_\lambda a \partial_\nu a}{a^2} + 2\eta^{\mu\lambda}\Gamma^\sigma_{\lambda\nu}\frac{\partial_\sigma a}{a}\right.\\ &\left. - \delta^\mu_\nu\left(\frac{R(\gamma)}{2} + 2h^{\lambda\rho}\frac{\partial_\lambda \partial_\rho a}{a} - h^{\lambda\rho}\frac{\partial_\lambda a \partial_\rho a}{a^2} + 2\eta^{\lambda\rho}\Gamma^\sigma_{\lambda\rho}\frac{\partial_\sigma a}{a}\right)\right].\end{aligned} \tag{1}$$

where the Christoffel symbols $\Gamma^\sigma_{\lambda\rho}$, and the Ricci tensor and the Ricci scalar are all computed with respect to the metric $\gamma_{\mu\nu}$,

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}\left(\partial_\nu h^\mu_\rho + \partial_\rho h^\mu_\nu - \partial^\mu h_{\nu\rho}\right). \tag{2}$$

At linear order we can now compute the Ricci tensor and the Ricci scalar:

$$\begin{aligned}R^\mu_\nu(\gamma) &= \frac{1}{2}\left(\partial_\nu \partial_\lambda h^{\mu\lambda} + \partial^\mu \partial^\lambda h_{\nu\lambda} - \partial_\lambda \partial^\lambda h^\mu_\nu - \partial^\mu \partial_\nu h^\lambda_\lambda\right),\\ R(\gamma) &= \partial_\mu \partial_\nu h^{\mu\nu} - \partial_\mu \partial^\mu h^\nu_\nu.\end{aligned}$$

The time has come to impose the gauge: the Christoffel symbols become

$$\Gamma^0_{00} = -\Phi', \qquad \Gamma^0_{i0} = \partial_i \Phi, \qquad \Gamma^0_{ij} = \Psi' \delta_{ij}, \qquad \Gamma^i_{jk} = \delta_{jk}\partial^i \Psi - \delta^i_k \partial_j \Psi - \delta^i_j \partial_k \Psi;$$

and each term in (1) can be made explicit, thus recovering the desired components of the linearised Einstein tensor.

2. Linearised energy-momentum conservation

The linearized Einstein equations read as

$$\delta G^\mu_\nu = 8\pi G \delta T^\mu_\nu, \tag{3}$$

where δT^μ_ν is the linearized deviation of the stress-energy tensor from the homogeneous part. To recover its components, recall that the stress tensor of an ideal fluid is written as

$$T^\mu_\nu = (\hat{\rho} + \hat{p}) u^\mu u_\nu - \delta^\mu_\nu \hat{p}$$

where we can separate the homogeneous part from the small perturbations by setting

$$\hat{\rho} = \rho + \delta\rho, \qquad \hat{p} = p + \delta p, \qquad u^0 = \frac{1 + \delta u^0}{a}, \qquad u^i = \frac{v^i}{a}.$$

Recalling that at any perturbative level it must hold that $u^2 \equiv -1$,

$$(-1 + 2\Phi)\left(1 + \delta u^0\right)^2 + \mathcal{O}(v^i v^i) = -1 \implies \delta u^0 = -\Phi,$$

while the linear fluctuations of the spatial part accounts for the physical velocity of the fluid, and can be constrained by residual gauge freedom.
Given that, the linearized stress-energy tensor fluctuations can be written as

$$\delta T^0_0 = \delta\rho \qquad \delta T^0_i = -\left(\rho + p\right) \partial_i v \qquad \delta T^i_j = -\delta^i_j \delta p,$$

and substituting into 3 we have, for the ij-component

$$\frac{1}{a^2}\partial^i \partial_j \left(\Phi + \Psi\right) + \\ -\frac{2}{a^2}\delta^i_j \left[-\Psi'' + \frac{1}{2}\triangle\left(\Phi + \Psi\right) + \frac{a'}{a}\left(\Phi' - 2\Psi'\right) + \left(2\frac{a''}{a} - \frac{a'^2}{a^2}\right)\Phi\right] = 8\pi G \delta p_{tot} \delta^i_j$$

Since the traceful and the longitudinal component are independent, we concluide that the longitudinal part of the Einstein tensor must vanish, imposing

$$\Psi = -\Phi.$$

This simple relation is really a consequence of the working hypothesis of an ideal fluid, and might need to be relaxed in more general contexts. In this simple framework it allows us to immediately eliminate the potential Ψ from our equations, thus giving

$$\triangle\Phi - 3\frac{a'}{a}\Phi' - 3\frac{a'^2}{a^2}\Phi = 4\pi G a^2 \delta\rho_{tot},$$
$$\Phi' + \frac{a'}{a}\Phi = -4\pi G a^2 \left[(\rho + p)v\right]_{tot},$$
$$\Phi'' + 3\frac{a'}{a}\Phi' + \left(2\frac{a''}{a} - \frac{a'^2}{a^2}\right)\Phi = 4\pi G a^2 \delta p_{tot};$$

in the above, the label *tot* on the right-hand side of the equations, stands for the sum over all the components of the ideal fluid on the right hand side.
Finally, the conservation equations on the linearized stress-energy tensor simply look like

$$\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu - \tilde{\Gamma}^\mu_{\mu\lambda} T^\lambda_\nu - \tilde{\Gamma}^\lambda_{\mu\nu} T^\mu_\lambda = 0.$$

and by keeping the first order terms in its expansion (with the clarification that here the Christoffels need to be computed with the *original* metric $g_{\mu\nu}$ and not just its conformal equivalent), we get the linearized conservation equations

$$\delta\rho' + 3\frac{a'}{a}\left(\delta\rho + \delta p\right) + \left(\rho + p\right)\left(\triangle v - 3\Phi'\right) = 0,$$
$$\left[(\rho + p)v\right]' + 4\frac{a'}{a}(\rho + p)v + \delta p + (\rho + p)\Phi = 0.$$

3. Linearised energy-momentum conservation

4. Helicity basis tensors

The rotation of polarization vectors can be described by

$$\mathbf{e}^{(1)'} = \mathbf{e}^{(1)} \cos\alpha - \mathbf{e}^{(2)} \sin\alpha,$$
$$\mathbf{e}^{(2)'} = \mathbf{e}^{(1)} \sin\alpha + \mathbf{e}^{(2)} \cos\alpha$$

If one constructs the vectors

$$\mathbf{e}^{\pm} = \mathbf{e}^{(1)} \pm i\mathbf{e}^{(2)}, \tag{4}$$

then the straightforward calculation gives that $\mathbf{e}^{\pm'} = e^{\pm i\alpha}\mathbf{e}^{\pm}$. This means that the helicities of these vectors are ± 1.

The tensors $e_{i,j}^{(+)}$ and $e_{i,j}^{(\times)}$ are obviously symmetric, they are also traceless because vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ are unit and orthogonal. Now, inverting (4)

$$\mathbf{e}^{(1)} = \frac{1}{2}\left(\mathbf{e}^{+} + \mathbf{e}^{-}\right), \qquad \mathbf{e}^{(2)} = \frac{1}{2i}\left(\mathbf{e}^{+} - \mathbf{e}^{-}\right) \tag{5}$$

one can rewrite the tensors $e_{i,j}^{(+)}$ and $e_{i,j}^{(\times)}$ as

$$e_{i,j}^{(+)} = \frac{1}{2\sqrt{2}}\left(e_i^{+} e_j^{+} + e_i^{-} e_j^{-}\right)$$
$$e_{i,j}^{(\times)} = \frac{1}{2\sqrt{2}i}\left(e_i^{+} e_j^{+} - e_i^{-} e_j^{-}\right)$$

From this expression it is obvious that the desired vectors of helicities ± 2 are

$$e_{i,j}^{\pm 2} = e_{i,j}^{(+)} \pm i e_{i,j}^{(\times)} \tag{6}$$

All possible 2-tensors which one can construct can be expressed as a linear combination of $e_i^{(\alpha)} e_j^{(\beta)}$ where $\alpha, \beta \in 1, 2$. The dimension of such tensors is then equal to 4. Being traceless and symmetric gives 2 conditions. Since we have already found 2 traceless symmetric tensors all the others can only be their linear combination.

5. Dimension of helicity basis

Using the formula (5) from the previous exercise, the third symmetric combination of unit vectors can be writen as

$$\left(e_i^{(1)} e_j^{(1)} + e_i^{(2)} e_j^{(2)}\right) = \frac{1}{2\sqrt{2}}\left(e_i^{(+)} e_j^{(-)} + e_i^{(-)} e_j^{(+)}\right) \tag{7}$$

Thus, the helicity of such vector is zero. Now one can construct a tensor

$$h_{i,j}^{T} = \delta_{i,j} - \frac{k_i k_j}{k^2} \tag{8}$$

and check that transversality condition holds:

$$k_i h_{i,j}^{T} = k_j h_{i,j}^{T} = 0$$

The trace of such tensor is equal to

$$h_{i,i}^{T} = 2$$

thus it is the desired tensor.