RELATIVITY AND COSMOLOGY II

Solutions to Problem Set 11 10th May 2024

1. Jeans instabilities

1. We substitute

$$\rho = \rho_0 + \delta\rho\ , \qquad p = p_0 + \delta p\ ,$$
$$\vec{v} = \delta\vec{v}\ , \qquad \phi = \phi_0 + \delta\phi\ .$$

in the system of equations and keep only the terms linear in the perturbations. We get

$$\nabla^2\delta\phi = 4\pi G\delta\rho \tag{1}$$

$$\frac{\partial\delta\rho}{\partial t} + \rho_0\vec{\nabla}\cdot\delta\vec{v} = 0\ , \tag{2}$$

$$\frac{\partial\delta\vec{v}}{\partial t} + \frac{v_s^2}{\rho_0}\vec{\nabla}\delta p + \vec{\nabla}\delta\phi = 0\ , \tag{3}$$

where we used $\delta p = v_s^2\delta\rho$. Taking the divergence of eq. (3) we find

$$\frac{\partial}{\partial t}\left(\rho_0\vec{\nabla}\delta\vec{v}\right) + v_s^2\nabla^2\delta\rho + \rho_0\nabla^2\delta\phi = 0\ . \tag{4}$$

Using eqs. (2) and (1), the above can be written as a second order differential equation for the density perturbation $\delta\rho$:

$$\frac{\partial^2\delta\rho}{\partial t^2} - v_s^2\nabla^2\delta\rho = 4\pi G\rho_0\delta\rho\ . \tag{5}$$

The solutions to this equation are plane waves

$$\delta\rho \propto e^{-i(\omega t - kz)}\ ,$$

where we assumed propagation in the $z-$direction only. In order to find the dispersion relation, we plug the above into eq. (5) to find (after a short calculation)

$$\omega(k)^2 = v_s^2k^2 - 4\pi G\rho_0\ .$$

Let us comment a bit on this result. The behaviour of the solutions depends on the wavenumber k. More precisely, it depends on whether k is larger or smaller than the critical value (Jeans wavenumber)

$$k_J = \left(\frac{4\pi G\rho_0}{v_s^2}\right)^{1/2}\ .$$

For $k > k_J$, ω is real so the perturbations oscillate as sound waves. On the other hand, for $k < k_J$ ω is imaginary and as a result the perturbations are amplified exponentially.

2. The system of equations describing the motion of the fluid is

$$\nabla^2\phi = 4\pi G\rho \ , \tag{6}$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}(\rho\vec{v}) = 0 \ , \tag{7}$$

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\phi \ . \tag{8}$$

We now consider perturbations on top of

$$\rho_0 \propto a^{-3} \ , \vec{v_0} = H\vec{r} \ , \phi_0 = \frac{2\pi G\rho_0 r^2}{3} \ .$$

Poisson equation immediately gives us

$$\nabla^2\delta\phi = 4\pi G\delta\rho \ .$$

Consider now eq. (7). Keeping terms linear in perturbations we find

$$\frac{\partial\delta\rho}{\partial t} + \rho_0\vec{\nabla}\cdot\delta\vec{v} + \delta\rho\left(\vec{\nabla}\cdot\vec{v_0}\right) + \left(\vec{v_0}\cdot\vec{\nabla}\right)\delta\rho = 0 \ .$$

Using $\vec{v_0} = H\vec{r}$ the above yields

$$\frac{\partial\delta\rho}{\partial t} + \rho_0(\vec{\nabla}\cdot\delta\vec{v}) + 3H\delta\rho + H(\vec{r}\cdot\vec{\nabla})\delta\rho = 0 \ .$$

Finally, eq. (8) becomes

$$\frac{\partial}{\partial t}(\vec{v_0}+\delta\vec{v}) + ((\vec{v_0}+\delta\vec{v})\cdot\vec{\nabla})\cdot(\vec{v_0}+\delta\vec{v}) = -\frac{1}{\rho_0\,(1+\delta\rho/\rho_0)}\vec{\nabla}\delta p - \vec{\nabla}(\phi_0+\delta\phi) \ .$$

Once again we are interested in the terms which are linear in the perturbations, therefore the above becomes

$$\frac{\partial\delta\vec{v}}{\partial t} + (\vec{v_0}\cdot\vec{\nabla})\delta\vec{v} + (\delta\vec{v}\cdot\vec{\nabla})\cdot\vec{v_0} = -\frac{1}{\rho_0}\vec{\nabla}\delta p - \vec{\nabla}\delta\phi = 0$$

Since $\delta p = v_s^2\delta\rho$ and $\vec{v_0} = H\vec{r}$, we obtain

$$\frac{\partial\delta\vec{v}}{\partial t} + H\delta\vec{v} + H(\vec{r}\cdot\vec{\nabla})\delta\vec{v} + \frac{v_s^2}{\rho_0}\vec{\nabla}\delta\rho + \vec{\nabla}\delta\phi = 0 \ .$$

2. Linear sizes of perturbations and masses of objects

The total mass of the object is given by

$$M(R) \simeq \frac{4\pi}{3}R^3\rho_{M,0}, \tag{9}$$

where $\rho_{M,0} = \Omega_M\rho_c$ is the present average mass density of dark matter together with baryons. The critical density $\rho_c = 1.5\cdot 10^{11}\cdot M_\odot/\text{Mpc}^3$, $\Omega_M = 0.27$. We find from Eq.(9) that

$$\begin{array}{l} R \sim (1-3)\text{Mpc} \Longleftrightarrow M \sim (10^{11} - 4\cdot 10^{12})M_\odot \\ R \sim (10-30)\text{Mpc} \Longleftrightarrow M \sim (10^{14} - 4\cdot 10^{15})M_\odot \\ R \sim (40-400)\text{kpc} \Longleftrightarrow M \sim (10^{7} - \cdot 10^{10})M_\odot \\ R \sim 10\text{kpc} \Longleftrightarrow M \sim 10^5 M_\odot. \end{array} \tag{10}$$

3. Free streaming length

The physical distance traveled by a generic free particle would be

$$\ell = R(t_{Eq}) \int_{t_0}^{t_{Eq}} v(t') \frac{\mathrm{d}t'}{R(t')} \tag{11}$$

Let's make an approximation by splitting that integral into part when it moves with ultrarelativistic speed $v \approx 1$ and nonrelativistic $v \approx p/m$.

$$\ell = R(t_{Eq}) \left[\int_{t_0}^{t_{NR}} \frac{\mathrm{d}t'}{R(t')} + \int_{t_{NR}}^{t_{Eq}} \frac{p(t')}{m} \frac{\mathrm{d}t'}{R(t')} \right] \tag{12}$$

Consider the second integral. Particle's momentum gets redshifted with the expansion of the universe, as usually, $p(t') \sim R(t')^{-1}$. Thus, $p(t') = \frac{R(t_{NR})}{R(t')} \cdot p(t_{NR}) \sim \frac{R(t_{NR})}{R(t')} \cdot m$, as the the transition between non-relativistic and relativistic regime happens when momentum is roughly of the scale of particle's mass.
This leads to

$$\ell = R(t_{Eq}) \left[\int_{t_0}^{t_{NR}} \frac{\mathrm{d}t'}{R(t')} + \int_{t_{NR}}^{t_{Eq}} \frac{R(t_{NR})\, \mathrm{d}t'}{R(t')^2} \right] \tag{13}$$

In a radiation-dominated universe $R(t) = \alpha t^{1/2}$, which allows to evaluate the integrals

$$\ell = \alpha t_{Eq}^{1/2} \left[\int_{t_0}^{t_{NR}} \frac{\mathrm{d}t'}{\alpha t'^{1/2}} + \int_{t_{NR}}^{t_{Eq}} \frac{\alpha t_{NR}^{1/2}\, \mathrm{d}t'}{\alpha^2 t'} \right] = 2(\sqrt{t_{Eq} t_{NR}} - \sqrt{t_0 t_{Eq}}) + \sqrt{t_{Eq} t_{NR}} \log \frac{t_{Eq}}{t_{NR}} \tag{14}$$

The time of emission, as it's in the Early Universe, may safely be approximated to be 0. That leads to the clean solution of

$$\ell = \sqrt{t_{Eq} t_{NR}} (2 + \log \frac{t_{Eq}}{t_{NR}}) \tag{15}$$

As the temperature and cosmic time during radiation dominance are related as

$$t = \frac{M_{pl}}{1.66 (g^*)^{1/2} T^2} \sim M_{pl}/T^2 \tag{16}$$

where the numerical factors have been neglected, as this is only order-of-magnitude-calculation (note that different decoupled species may have different temperature).
Assembling it all together:

$$\ell \simeq \frac{M_{pl}}{T_{Eq} T_{NR}} (2 + 2 \log \frac{T_{NR}}{T_{EQ}}) \tag{17}$$

with $T_{NR} \sim m = 0.1\,\mathrm{eV}$ and $T_{EQ} = 0.7\,\mathrm{eV}$, so the entire bracket content is of order of unity. Neglecting it and cleaning up the units leads to

$$\ell \sim \ell_{pl} \frac{M_{pl}^2}{T_{Eq} T_{NR}} \sim 10^{22}\,\mathrm{m} \tag{18}$$

The matter density at t_{Eq} is evaluated from current densities

$$\rho_{Eq} = (\Omega_M \rho_c) \cdot (1 + z_{Eq})^3 \sim 10^{-16}\,\mathrm{kg/m^3} \tag{19}$$

Finally, within a sphere of such length there is roughly a mass of

$$M = \frac{4}{3}\pi\rho_{Eq}\ell^3 \sim 10^{50}\,\text{kg} \sim 10^{19} M_\odot \tag{20}$$

That places the large scale structure of neutrinos on a scale way bigger than galaxy cluster. Such matter is 'too hot' (= perturbations too smeared out) to form structure responsible for galaxy rotation curves.