RELATIVITY AND COSMOLOGY II

Solutions to Problem Set 10 3rd May 2024

1. Sphalerons in the Standard Model and beyond

As given in the exercise, the following quantities are conserved in a sphaleron transition:

$$\Delta_e \equiv L_e - \frac{1}{3}B, \qquad \Delta_\mu \equiv L_\mu - \frac{1}{3}B, \qquad \Delta_\tau \equiv L_\tau - \frac{1}{3}B. \tag{1}$$

Here B is the total baryon number (again, number of baryons minus number of antibaryons), while the lepton number L (number of leptons minus number of antileptons) has been split in three contributions from the three different generations, with $L \equiv L_e + L_\mu + L_\tau$.

Adding the three conserved quantities above gives a fourth, more familiar, conserved quantity:

$$\Delta_e + \Delta_\mu + \Delta_\tau = L_e + L_\nu + L_\tau - B = L - B. \tag{2}$$

So sphaleron reactions can violate both B and L, but in such a way that $B - L$ remains conserved.

Now we want to say something about total baryon number B and total lepton number L. The difference of fermion and antifermion number density is

$$n_f - n_{\bar{f}} = \mu \frac{gT^2}{6}. \tag{3}$$

The net baryon number B of the standard model is caused by 36 quark degrees of freedom (2 quarks with 3 colors each per family, 3 families, left- or right handed) that each carry individual baryon number of 1/3, so we get $B = 2\mu_q T^2$. The net lepton number (per family) is caused by 3 lepton degrees of freedom (left and right electron, left neutrino) that each carry individual lepton number 1, so we get $L_i = \frac{1}{2}\mu_i T^2$. Combining all that gives

$$\begin{aligned} B - L &= -L_e - L_\mu - L_\tau + B \\ &= -\frac{1}{2}\left(\mu_e + \mu_\mu + \mu_\tau\right) T^2 + 2\mu_q T^2. \end{aligned} \tag{4}$$

1. Sphaleron processes convert 3 baryons (9 quarks) into 3 antileptons. In thermal equilibrium we then have

$$9\mu_q = -\left(\mu_e + \mu_\mu + \mu_\tau\right). \tag{5}$$

   Putting everything together, we can relate the created baryon asymmetry B as a function of the conserved quantity $B - L$, for sphaleron transitions in the Standard Model:

$$\begin{aligned} B - L &= \frac{1}{2} \cdot 9\mu_q T^2 + 2\mu_q T^2 \\ &= \frac{13}{2}\mu_q T^2 \\ &= \frac{13}{4} B\ . \end{aligned} \tag{6}$$

2. As X can decay to ud ($B = \frac{2}{3}$, $L = 0$), to $\bar{u}\bar{l}$ ($B = -\frac{1}{3}$, $L = -1$) and to $\bar{d}\nu$ ($B = -\frac{1}{3}$, $L = 1$), the decay of X results in average change of

$$(B - L)_X = \frac{2}{3} r_{ud} + \frac{2}{3} r_{\bar{u}\bar{l}} - \frac{4}{3} r_{\bar{d}\nu}, \tag{7}$$

where

$$r_{ud} = \frac{\Gamma_{ud}}{\Gamma_{X,\mathrm{tot}}}, \quad r_{\bar{u}\bar{l}} = \frac{\Gamma_{\bar{u}\bar{l}}}{\Gamma_{X,\mathrm{tot}}}, \quad r_{\bar{d}\nu} = \frac{\Gamma_{\bar{d}\nu}}{\Gamma_{X,\mathrm{tot}}} \tag{8}$$

are branching ratios. As $r_{ud} + r_{\bar{u}\bar{l}} + r_{\bar{d}\nu} = 1$,

$$(B - L)_X = \frac{2}{3}(1 - r_{\bar{d}\nu}) - \frac{4}{3} r_{\bar{d}\nu} = \frac{2}{3} - 2 r_{\bar{d}\nu} \tag{9}$$

Similarly, the decay of $\bar{X}$ results in

$$(B - L)_{\bar{X}} = -\frac{2}{3} + 2\tilde{r}_{d\bar{\nu}} \quad \text{with } \tilde{r}_{d\bar{\nu}} = \frac{\tilde{\Gamma}_{d\bar{\nu}}}{\tilde{\Gamma}_{\bar{X},\mathrm{tot}}} . \tag{10}$$

Note that $\tilde{\Gamma}_{\bar{X},\mathrm{tot}} = \Gamma_{X,\mathrm{tot}}$ is required by CPT invariance.

3. Considering decay of equal populations of X and $\bar{X}$:

$$(B - L)_{\mathrm{tot}} = n_X \cdot \left[(B - L)_X + (B - L)_{\bar{X}} \right] = 2 n_X \cdot (\tilde{r}_{d\bar{\nu}} - r_{\bar{d}\nu}) . \tag{11}$$

Therefore, if $\tilde{r}_{d\bar{\nu}} \neq r_{\bar{d}\nu}$, so the process is not CP-invariant, net $B - L$ charge is produced, which results in non-zero total baryon number.

2. Rotation curves of galaxies

We start from

$$v^2 = \frac{GM(r)}{r} , \tag{12}$$

where for a spherically symmetric distribution

$$M(r) = 4\pi \int_0^r \mathrm{d}r' r'^2 \rho(r') , \tag{13}$$

with $\rho(r)$ the density.

1. We consider first the case where for $r < r_{\mathrm{core}}$ the density is constant, i.e.

$$\rho(r) = \begin{cases} \rho_0 & \text{for } r \leq r_{\mathrm{core}} \\ 0 & \text{for } r > r_{\mathrm{core}} \end{cases} , \tag{14}$$

where ρ_0 is core's density. Evaluating the integral gives

$$M(r) = \begin{cases} \frac{4}{3}\pi\rho_0 r^3 & \text{for } r \leq r_{\mathrm{core}} \\ \frac{4}{3}\pi\rho_0 r_{\mathrm{core}}^3 & \text{for } r > r_{\mathrm{core}} \end{cases} , \tag{15}$$

which results in

$$v \propto r^{-1/2} \tag{16}$$

outside the galaxy's center.

2. Consider now the possibility that the density obeys inverse power law, i.e.

$$\rho \propto r^{-\alpha} \ , \ \alpha > 0 \ . \tag{17}$$

In this case, (13) gives us

$$M(r) \propto r^{3-\alpha} \ , \ \alpha < 3 \ . \tag{18}$$

Therefore, the circular velocity becomes

$$v \propto r^{1-\alpha/2} \ . \tag{19}$$

To get flat rotational curve, $v \approx$ const., one needs $\alpha = 2$.

3. Potential well

1. We start by solving exactly the three-dimensional problem. Schrödinger's equation is given by

$$-\frac{\vec{\nabla}^2}{2m}\Psi = E\Psi \ , \tag{20}$$

where m is the mass of the particle, Ψ its wave-function that must be continuous everywhere and vanish outside the cube. The solutions inside the cube are

$$\Psi_{abc} = \sin\left(a\frac{\pi x}{R}\right)\sin\left(b\frac{\pi y}{R}\right)\sin\left(c\frac{\pi z}{R}\right) \ , \tag{21}$$

where a, b and c are positive integers. Plugging the above into Schrödinger's equation we find the corresponding energy eigenvalues

$$E_{abc} = \frac{\pi^2}{2mR^2}\left(a^2+b^2+c^2\right) \ . \tag{22}$$

To compute the number of states of momentum below p_{max}, one requires total energy to stay below $E_{\text{max}} = \frac{p_{\text{max}}^2}{2m}$. This results in condition

$$\frac{\pi^2}{2mR^2}\left(a^2+b^2+c^2\right) < \frac{p_{\text{max}}^2}{2m} \tag{23}$$

which is description of a sphere of radius $r = p_{\text{max}}R/\pi$ in (a, b, c)-space.

The number of available states therefore is proportional to sphere's volume

$$N \approx \frac{1}{8}\cdot\frac{4\pi}{3}r^3 = \frac{1}{6\pi^2}p_{\text{max}}^3R^3, \tag{24}$$

where the approximation holds if $r \gg 1$.

2. In the course we saw that, via classical approximation,

$$N_{\text{approx.}} \approx \frac{1}{(2\pi)^3}\int \mathrm{d}^3p\ \mathrm{d}^3x = \frac{1}{(2\pi)^3}\frac{4\pi}{3}p_{\text{max}}^3R^3. \tag{25}$$

This is exactly the same value as (24)

Since there are 3 neutrinos and 3 antineutrinos, we can put $6N$ neutrinos of negative or zero energy in these levels.

Note: It shouldn't come as a surprise, as both formulas are classical approaches to the same problem, only the limit was taken at different points. Both questions can be expressed as what is the partition function of system with Hamiltonian

$$H(x,p) = \begin{cases} \frac{p^2}{2m} & \text{if } p < p_{\max} \text{ and } x \text{ is inside the cube,} \\ \infty & \text{otherwise.} \end{cases}$$

One can either find the states quantum mechanically and evaluate $Z(\beta) = \sum e^{-\beta E_i}$ or compute the classical partition function for a particle $Z(\beta) = \frac{1}{(2\pi\hbar)^3} \int e^{-\beta H(x,p)}\, \mathrm{d}^3x\, \mathrm{d}^3p$ in the limit of $\beta \to 0$. The classical limit in the first case is taken via replacing the number of points in a sphere with a volume integral.

4. Primordial Black Holes

Hawking temperature simply states that black holes can radiate and emit energy (and particles) as if it is an object of temperature $\frac{1}{8\pi GM}$. According to Stefan- Boltzmann law,

$$P(T) = \sigma_0 A T^4, \tag{26}$$

where σ_0 is Stefan–Boltzmann constant, $\sigma_0 = \frac{1}{60}\pi^2$ in the Planck units $\hbar = c = G = 1$. The surface from which a black hole emits energy is the event horizon of the black hole. In case of Schwarzschild black holes, as we know from Relativity and Cosmology I, the horizon is spherically symmetric and appears at the radius $r_s = 2M$. This gives the total area of emission

$$A = 4\pi r_s^2\ . \tag{27}$$

Combining the two results above and also the expression for Hawking temperature, we can write down the total power of energy emission of a Schwarzschild black hole,

$$P = \frac{4\pi^3}{15} M^2 T^4 = \frac{1}{15360\pi M^2}\ . \tag{28}$$

However, P is nothing but $-\frac{dM}{dt}$ since for a static black hole the total energy it possesses only comes from its mass. Thus, we have a differential equation to solve for the life time of a black hole; as the mass of a black hole reduces to zero, the black hole evaporates and the time it takes to do so will be the life time of this black hole. Solving the differential equation

$$\frac{dM}{dt} = -\frac{1}{15360\pi M^2} \tag{29}$$

yields

$$M = \left(\frac{\tau}{5120\pi}\right)^{1/3}, \tag{30}$$

where τ stands for the life time. Let's restore SI units:

$$M = M_{\mathrm{Pl}} \left(\frac{1}{5120\pi} \frac{\tau}{t_{\mathrm{Pl}}}\right)^{1/3} \tag{31}$$

where $M_{\mathrm{Pl}} = 2.17 \times 10^{-8}\,\mathrm{kg}$ is the Planck mass, and $t_{\mathrm{Pl}} = 5.39 \times 10^{-44}\,\mathrm{s}$ the Planck time. For primordial BH to survive till today, τ must be greater than current age of the universe, $4.3 \times 10^{17}\,\mathrm{s}$, the mass of the black hole has to be at least

$$M_{\min} = 1.7 \times 10^{11}\,\mathrm{kg} \tag{32}$$

Scientifically speaking, this is roughly 0.8 of the mass of trash United States produces every year (2.4×10^{11} kg).
The mass here suggests that the Hawking temperature of this black hole is

$$T_{\mathrm{max}} = \frac{1}{8\pi G M_{\mathrm{min}}} \approx 59\,\mathrm{MeV} \tag{33}$$

which is higher than the mass of electrons and neutrinos. This means that the black hole could also emit these particles from the beginning, energy emission must include not only production of photons (which we have considered from the beginning), but also neutrinos, electrons, and gravitons (which we should consider because the gravitational field is strong around the black hole). Therefore,

$$P' = \frac{1}{2}\Big(\underbrace{\;2\;}_{\text{photons}} + \underbrace{\frac{7}{8}(4+6)}_{\text{leptons}} + \underbrace{\;2\;}_{\text{gravitons}}\Big)P = \frac{51}{8}P. \tag{34}$$

The factor $\frac{1}{2}$ refers to 2 degrees of freedom of the photons we considered previously. Inside the bracket we have 2 degrees of freedom for photons and 2 degrees of freedom for gravitons. The $4+6$ in the bracket is related to electrons, positrons (both helicities) and three types of neutrinos and antineutrinos (only left-handed). This modified power of radiation changes the minimum mass required for a primordial black hole to survive until present day:

$$M' = M\left(\frac{51}{8}\right)^{1/3} \approx 3.2 \times 10^{11}\,\mathrm{kg}. \tag{35}$$

This is roughly equal to the mass of all living humans today (3.9×10^{11} kg).
These are two limits of lifetime of a black hole. To get the exact answer one must include “gray factors”, taking propagation of emitted particles into account. However precise considerations of this type are beyond the scope of this course.