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Relativity and cosmology I

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1 SPECIAL RELATIVITY

1.1 REMINDERS

1.1.1 Natural systems of units and Einstein convention

For historical reasons, the most widely used system of units today is the Centimeter-Gram-Second (CGS) system and the International System (SI). In the CGS system, for instance, there are 4 fundamental units (cm, g, s, K) and 4 fundamental constants:

speed of light:	$c = 29\,979\,245\,800\,\text{cm}\,\text{s}^{-1}$
Planck constant:	$\hbar = 1{,}054 \times 10^{-27}\text{erg s}$
Newtonian constant of gravity:	$G = 6{,}67\,\text{cm}^3\,\text{g}^{-1}\,\text{s}^{-2}$
Boltzmann constant:	$k_B = 1{,}38 \times 10^{-16}$ erg K^{-1}

where erg $= gcm^2s^{-2}$. To simplify their equations, some physicists, especially in High Energy Physics (HEP), prefer to work with a system of units where part or all of these fundamental constants are set to 1. Since such systems of units are defined directly in terms of the fundamental constants of nature, they are often called *natural systems of units*. The two most commonly used systems are

- The extreme choice: $c = \hbar = G = k_B = 1$.
  All quantities are dimensionless in this system. It is typically used in theories of quantum gravity, e.g. string theory.

- The HEP choice: $c = \hbar = k_B = 1$.
  Having set three fundamental constants to 1, we are left with one physical dimension which is shared by mass, temperature, inverse length, and inverse time. The unit of the remaining dimension is conventionally chosen to be 1 GeV $= 10^9$ eV. Conversion factors relating the HEP units to the CGS units are shown in Table 1.

Throughout this course, we will be using the HEP choice.

Another convention which makes life easier when one manipulates complicated expressions wit a lot of indices is the one introduced by Einstein, which we will follow here. Whenever in an expression the same index appears twice, once up and once down (i.e. μ in $A_{\alpha\beta\mu}B^{\mu}_{\rho\delta}$), this index is understood to be summed over. One often calls such indices dummy indices.

Quantity	Conversion HEP quantity
Length	$\text{GeV}^{-1} = 1{,}98 \times 10^{-14}$ cm
Time	$\text{GeV}^{-1} = 6{,}58 \times 10^{-25}$ s
Mass	$\text{GeV} = 1{,}78 \times 10^{-24}$ g
Temperature	$\text{GeV} = 1{,}16 \times 10^{13}$ K

Table 1: HEP to CGS system of units conversion.

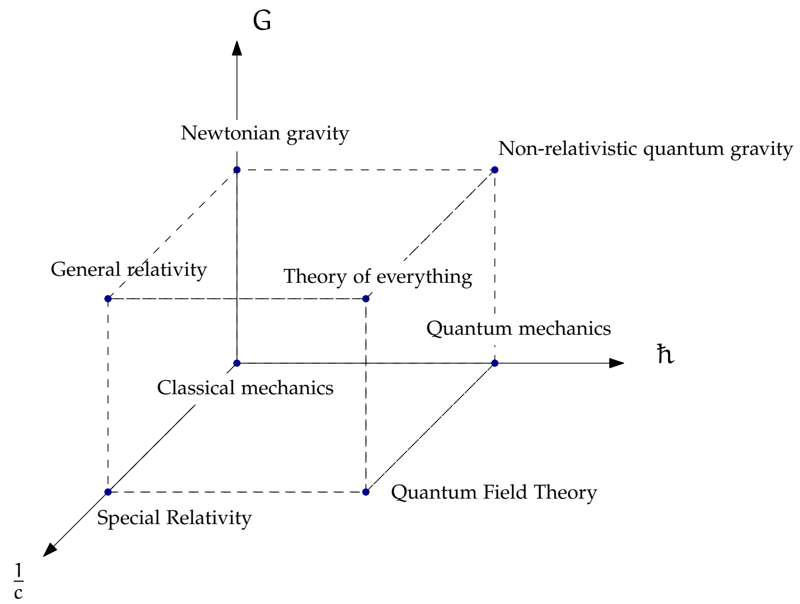

Figure 1: The cube of physics.

1.1.2 The cube of physics

In 1928, to illustrate how the different fields of physics fit together, Gamow and Ivanenko introduced the cube of physics (see Fig. 1). It shows a coordinate system with $1/c$, $\hbar$, and G as its axes, where different fields of physics lie:

- At $(0, \hbar, 0)$ lies non-relativistic Quantum Mechanics,
- At $(\frac{1}{c}, 0, 0)$ lies Special Relativity,
- At $(\frac{1}{c}, \hbar, 0)$ lies Quantum Field Theory.

Although strictly speaking c, G, and $\hbar$ are constants, here we imagine varying them to certain limiting values to see how the laws of physics look like in that limit. We should keep in mind that this practice of varying the fundamental constants is not to be taken too literally. When we say $1/c \to 0$, we actually mean that we are looking at a physical system in which the particles' velocities are extremely small compared to the speed of light.

1.1.3 Symmetries, Newtonian mechanics and Maxwellian electrodynamics

Here we briefly review Newtonian mechanics and Maxwellian electrodynamics, putting the emphasis on the symmetries they respect. In particular, we recall how comparing their symmetries naturally hints towards special relativity.

Consider a system of N massive particles interacting with a central potential $U(\mathbf{x})$. The dynamics of this system is described by the Hamiltonian

$$H = \sum_i \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} U(|\mathbf{x}_i - \mathbf{x}_j|) \tag{1.1}$$

from which we can derive the equations of motion

$$m \frac{d^2 \mathbf{x}_i}{dt^2} = -\frac{\partial U}{\partial \mathbf{x}_i} \quad \forall i \tag{1.2}$$

One can check that the form of the above equations is left unchanged under the following transformations:

- Translation: $\mathbf{x}_i \to \mathbf{x}_i + \mathbf{a}$.
  i.e. space is homogeneous.
- Rotation: $(\mathbf{x}_i)^\alpha \to O^\alpha_\beta\,(\mathbf{x}_i)^\beta$ with $OO^T = 1$
  i.e. space is isotropic.
- Galilean transformation: $\mathbf{x}^i \to \mathbf{x}^i + \mathbf{v_0}t, \qquad t \to t$
  i.e. physics does not depend on our choice of inertial reference frame and time is absolute.

If spacetime is symmetric under a Galilean transformation then a particle whose velocity is $\mathbf{v}$ in one inertial reference frame has a velocity

$$\mathbf{v}' = \mathbf{v} - \mathbf{v}_0 \tag{1.3}$$

in a different inertial reference frame which is moving with velocity $\mathbf{v}_0$ relative to the initial reference frame. We refer to this as the Galilean law of velocity transformation.

Let us now turn our attention to Maxwell's equations

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho \qquad & -\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} &= \mathbf{j} \\ \nabla \cdot \mathbf{B} &= 0 \qquad & \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} &= 0 \end{aligned} \tag{1.4}$$

Maxwell's equations are invariant under translations. But, unlike the Newtonian equations of motion, they are not invariant under Galilean transformations; they are instead invariant under Lorentz transformations[1]. Gallilean transformation are replaced by Lorentz boosts. For example, a Lorentz boost in the x-direction transforms the spacetime coordinates as

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\Lambda^\mu{}_\nu} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{1.5}$$

The charge density and currents transform in a similar way. Electric and magnetic field transform in a slightly more complicated way, see later.

In particular, the simple Gallilean composition law for the velocities (1.3) becomes

$$v' = \frac{v - v_0}{1 - v v_0} \tag{1.6}$$

when an inertial reference frame is moving with velocity v_0 in x-direction relative to the initial reference frame. Note that a general Lorentz transformation is parameterized by 6 numbers (3 boosts and 3 rotations). The combination of a Lorentz transformation and a translation is called a Poincaré transformation.

We are now faced with a conundrum. On one hand, Newton's equations of motion are invariant under Galilean transformations but not under Lorentz transformations. On the other hand, Maxwell's equations are Lorentz invariant but not Galilean invariant. In order to determine which of the two symmetries, Galilean or Lorentzian, is more fundamental, we need to resort to experiments. A key observable is the speed of light, as (1.3) predicts it is frame dependent whilst (1.6) predicts it to be frame invariant.

The famous Michelson-Morley experiment attempted to measure this potential dependence of the speed of light on the velocity of the observer, but found no such dependence, thus favouring the Lorentz symmetry over Galilean symmetry. As

[1] Rotations are a subgroup of Lorentz transformations

of today, a countless number of experiments has confirmed this result. This has led us to modifying the laws of motion to that of special relativity, where time is not absolute and the speed of light is frame-independent. The discovery of special relativity, however, does not undo the success of Newtonian mechanics in the low velocity regime. It merely clarifies the scope of Newtonian mechanics and goes beyond it.

1.1.4 Lorentz Tensor

To effortlessly make sure that our equations are Lorentz invariant, we express them in terms of Lorentz tensors. Lorentz tensors are objects which transform in a well-defined way under Lorentz transformations. The simplest non-trivial Lorentz tensors are contravariant and covariant four-vectors. A contravariant 4-vector is a set of 4 quantities v^μ which transform under a Lorentz transformation as

$$v^\mu \to v'^\mu = \Lambda^\mu{}_\nu v^\nu \quad \text{for} \quad \mu, \nu = 0, 1, 2, 3 \tag{1.7}$$

with repeated indices summed over and $\Lambda^\mu{}_\nu$ being any matrix satisfying

$$\eta_{\mu\nu} = \eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \tag{1.8}$$

and

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.9}$$

is the Minkowski metric. A covariant 4-vector is a set of 4 quantities w_μ which transform under Lorentz transformations as

$$w_\mu \to w'_\mu = \Lambda_\mu{}^\nu w_\nu \quad \text{for} \quad \mu, \nu = 0, 1, 2, 3 \tag{1.10}$$

where $\Lambda_\mu{}^\nu$ is a shorthand notation for $\left((\Lambda^\mu{}_\nu)^{-1}\right)^T$ which is in general different from $\Lambda^\mu{}_\nu$. An example of a contravariant 4-vector would be the 4-velocity $u^\mu = dx^\mu/d\tau$ and an example of a covariant vector would be the derivative $\partial_\mu S \equiv \partial S/\partial x^\mu$ of some function S. A covariant 4-vector v_μ can be constructed from a contravariant 4-vector v^μ with the help of the Minkowski metric

$$v_\mu = \eta_{\mu\nu} v^\nu . \tag{1.11}$$

Conversely, a contravariant 4-vector w^μ can be constructed from a covariant 4-vector w_μ as follows

$$w^\mu = \eta^{\mu\nu} w_\mu \tag{1.12}$$

where $\eta^{\mu\nu}$ is the inverse matrix of $\eta_{\mu\nu}$. A contravariant 4-vector v^μ can be contracted with a covariant 4-vector w_μ to give a scalar S (a quantity that is invariant under Lorentz transformations)

$$S = v^\mu w_\mu . \tag{1.13}$$

A *Lorentz tensor* is a set of quantities, written compactly as a symbol with (in general) both covariant and contravariant indices $T^{\mu\nu\cdots}{}_{\alpha\beta\ldots}$, which transform under a Lorentz transformation as

$$T^{\mu\nu\cdots}{}_{\alpha\beta\ldots} \longrightarrow T^{\mu\nu\cdots}{}_{\alpha\beta\ldots} = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} \cdots \Lambda_\alpha{}^{\alpha'} \Lambda_\beta{}^{\beta'} T^{\mu'\nu'}{}_{\alpha'\beta'} \tag{1.14}$$

Invariant Lorentz tensors/pseudotensors, i.e. tensor-indices carrying objects that remain unchanged under Lorentz transformations, play important roles in physics. Other than the Minkowski metric $\eta_{\mu\nu}$, which is invariant under Lorentz transformations by definition (see (1.8)), there are two more invariant tensors/pseudotensors:

$$\delta^{\nu}_{\mu} = \eta_{\mu\alpha}\eta^{\alpha\nu} = \text{diag}(1, 1, 1, 1) \tag{1.15}$$

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of 0123} \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of 0123} \\ 0 & \text{otherwise.} \end{cases} \tag{1.16}$$

It is important to understand that not every object with tensor indices is automatically a tensor. For example, $\epsilon_{\mu\nu\rho\sigma}$ carries tensor indices, but it does not transform as a tensor. For another example, although the Lorentz transformation matrix $\Lambda^{\mu}{}_{\nu}$ carries tensor indices, it makes no sense to say that $\Lambda^{\mu}{}_{\nu}$ is a tensor because it represents a transformation and not a coordinate-dependent set of quantities.

1.1.5 Covariant formulation of Maxwellian electrodynamics and Newtonian mechanics

Armed with the knowledge of tensors, we can reformulate Maxwellian electrodynamics and Newtonian mechanics in terms of manifestly Lorentz-covariant equations, i.e. equations whose left-hand side and right-hand side transform in the same way under Lorentz transformations.

In order to write the Maxwell equations in a covariant form, we introduce an antisymmetric *field-strength tensor*

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.17}$$

with A_ν the potential four-vector

$$A_\nu = \begin{pmatrix} \varphi \\ \mathbf{A} \end{pmatrix} \tag{1.18}$$

where ϕ and $\mathbf{A}$ are the scalar and vector potential. Note that $F_{\mu\nu}$ is a gauge-invariant quantity, meaning that it remains unchanged under a gauge transformation

$$A_\nu \longrightarrow A_\nu{}' = A_\nu - \partial_\nu \alpha.$$

When expressed in terms of $F_{\mu\nu}$, the four Maxwell equations reduce to two covariant equations

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{1.19}$$

$$\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 0 \tag{1.20}$$

where j^ν is the four-current

$$j^\nu = \begin{pmatrix} \rho \\ \mathbf{j} \end{pmatrix}. \tag{1.21}$$

Recall that in special relativity the spacetime interval between two events is defined as

$$ds^2 = \eta_{\mu\nu} dx^\nu dx^\mu = dt^2 - (dx^2 + dy^2 + dz^2) \tag{1.22}$$

which is invariant under Lorentz transformations. It defines the causal structure of the spacetime interval. Intervals split up in three categories

- Time-like: $ds^2 > 0$
- Light-like/null: $ds = 0$
- Space-like $ds^2 < 0$.

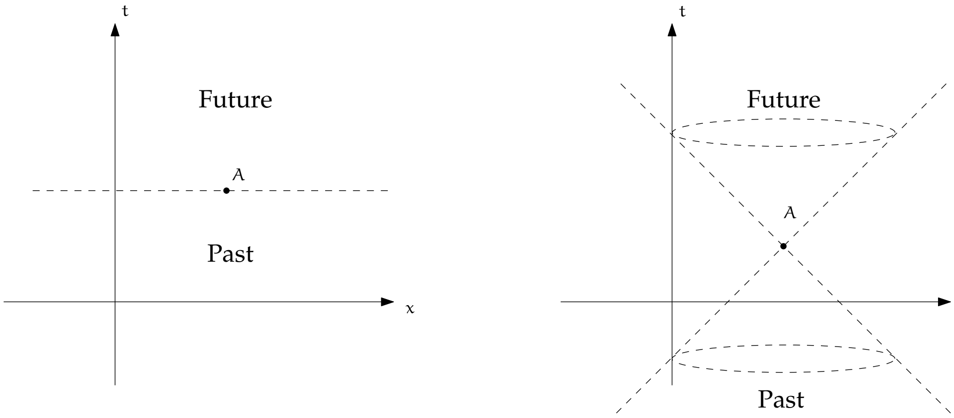

Figure 2: Spacetime regime that may influence particle A (labeled "past") and spacetime regime that may be influenced by particle A (labeled "future") in Newtonian mechanics (left) and special relativity (right).

While time-like separated points may influence one another (causally connected), space-like separated points are completely out of contact (causally disconnected). In the Δt-Δx plane of the two events, the set of all possible null intervals form future and past light cones. A particle may only influence particles inside its future lightcone and can only be influenced by particles inside its past lightcone, see figure 2.

For a particle moving with speed v, the spacetime interval is given by

$$ds^2 = dt^2 - v^2 dt^2 = \gamma^{-2} dt^2 \quad \longrightarrow \quad \frac{d}{ds} = \gamma \frac{d}{dt} \tag{1.23}$$

We can then define the 4-velocity and 4-acceleration as follows

$$u^\mu = \frac{dx^\mu}{ds} \tag{1.24}$$

$$a^\mu = \frac{d^2 x^\mu}{ds^2} = \frac{du^\mu}{ds} \tag{1.25}$$

and, from the 4-velocity u^μ, we can define the relativistic momentum as

$$p^\mu = m u^\mu \tag{1.26}$$

The relativistic equation of motion for a particle of mass m and charge q in an electromagnetic field can be written in a covariant form as

$$\frac{dp^\mu}{ds} = q F^{\mu\nu} u_\nu \tag{1.27}$$

or

$$m \frac{d^2 x^\mu}{ds^2} = q F^{\mu\nu} u_\nu \tag{1.28}$$

In the special case of constant and uniform electric field and zero magnetic field, the above equation describes a constant acceleration motion.

1.2 SPECIAL RELATIVITY IN ARBITRARY COORDINATE SYSTEMS

1.2.1 Coordinate transformation

While a Lorentz transformation is a linear transformation

$$x^\mu = \Lambda^\mu{}_\nu x^\nu \quad \longrightarrow \quad \text{linear}$$

a general coordinate transformation

$$x^\mu = f^\mu(x^\nu) \quad \longrightarrow \quad \text{non-linear in general}$$

is not necessarily linear. Therefore, it cannot be expressed as a simple matrix multiplication. To see this, take a look at the example below.

Example 1.1. Consider a charged particle in a constant and uniform electric field $\mathbf{E} = E\hat{x}$ and zero magnetic field. The particle obeys (1.28) which in this case reduces to

$$m\frac{d^2x^\mu}{ds^2} = qEu^\mu \tag{1.29}$$

whose $i = 1$ component (x component) is

$$\frac{d}{dt}\left[\frac{v}{\sqrt{1-v^2}}\right] = \frac{eE}{m} = a \tag{1.30}$$

where $v = dx^1/dt$. The solution to this equation is

$$v(t) = \frac{at}{\sqrt{1+a^2t^2}} \tag{1.31}$$

$$x(t) = x_0 + \frac{1}{a}\left[\sqrt{1+a^2t^2} - 1\right]. \tag{1.32}$$

Now, we would like to perform a coordinate transformation from the lab frame to the rest frame of the particle. We have just found that the space coordinate x' in the rest frame of the particle is related to the space coordinate in the lab frame x as

$$x' = x - \frac{1}{a}\left[\sqrt{1+a^2t^2} - 1\right]. \tag{1.33}$$

To do so, we need to relate the time measured in the rest frame of the particle τ, i.e. the proper time, to the time in the lab frame t. In the lab frame, we have

$$\begin{aligned} ds^2 &= dt^2 - dx^2 \\ &= dt^2(1-v^2). \end{aligned} \tag{1.34}$$

Since $dx^i = 0$ in the rest frame of the particle, we have

$$ds^2 = d\tau^2. \tag{1.35}$$

The invariance of the spacetime interval ds^2 gives

$$d\tau = \sqrt{dt^2 - v^2dt^2} = dt\sqrt{1 - \frac{a^2t^2}{1+a^2t^2}} = \frac{dt}{\sqrt{1+a^2t^2}}. \tag{1.36}$$

Integrating the above equation, we get

$$\tau = \frac{1}{a}\log\left(at + \sqrt{1+a^2t^2}\right). \tag{1.37}$$

Note that $\tau < t$, as we would expect from time dilation. Putting our results together, the coordinate transformation from the lab frame to the rest frame of the particle is given by

$$t \to t' = \tau = \frac{1}{a}\log\left(at + \sqrt{1+a^2t^2}\right) \tag{1.38}$$

$$x \to x' = x + \frac{1}{a}\left[\sqrt{1+a^2t^2} - 1\right] \tag{1.39}$$

As we can see, both transformations are non-linear. This should not come as a surprise, the more familiar transformations from Cartesian to cylindrical coordinates and from Cartesian to spherical coordinates are non-linear as well.

So far, we have mainly studied special relativity in Cartesian coordinates. However, in many cases it is not convenient to work in a Cartesian coordinate system. For instance, if the system under consideration has a spherical symmetry then it is more convenient to use spherical coordinates. This motivates the study of special relativity in arbitrary coordinate systems. We will adopt the following notations: latin indices are used to label Cartesian coordinates while Greek indices will denote coordinates in an arbitrary system.

In general, the Cartesian coordinates x^i can be expressed in terms of the coordinates x'^μ in an arbitrary system as

$$x^i = f^i(x^\mu), \tag{1.40}$$

or, in differential form

$$dx^i = \frac{\partial x^i}{\partial x'^\mu} dx'^\mu. \tag{1.41}$$

The transformation is one-to-one if the Jacobian

$$J = \frac{\partial x^i}{\partial x'^\mu} \tag{1.42}$$

has non-zero determinant. If we know how to write x^i in a Cartesian system in terms of x'^μ in an arbitrary system

$$x^i = x^i(x'^\mu), \quad dx^i = \frac{\partial x^i}{\partial x'^\mu} dx'^\mu, \tag{1.43}$$

and how to write the same x^i in terms of $\tilde{x}^\nu$ in another arbitrary system

$$x^i = x^i(\tilde{x}^\nu), \quad dx^i = \frac{\partial x^i}{\partial \tilde{x}^\nu} d\tilde{x}^\mu, \tag{1.44}$$

then we can work out a direct connection between the coordinates of the two arbitrary systems:

$$x^\mu = x^\mu(\tilde{x}^\nu), \quad dx^\mu = \frac{\partial x^\mu}{\partial \tilde{x}^\nu} d\tilde{x}^\nu. \tag{1.45}$$

In other words, this shows how coordinates transform when we go from one arbitrary coordinate system to another. One consequence of this is that the derivative of a scalar function ϕ transforms as

$$\frac{\partial \phi}{\partial x^\mu} = \frac{\partial \phi}{\partial x'^\nu} \frac{\partial x'^\nu}{\partial x^\mu}. \tag{1.46}$$

Previously, we have seen how contravariant and covariant Lorentz 4-vectors transform under Lorentz transformations. For a general coordinate transformation of which the Lorentz transformation is a special case, a contravariant 4-vector A^μ is defined as an object that transforms in the same way as dx^μ under coordinate transformations, i.e.

$$A^\mu = \frac{\partial x^\mu}{\partial x'^\nu} A'^\nu \tag{1.47}$$

and a covariant four-vector A_μ is defined as an object that transforms in the same way as the derivative of a scalar function $\partial_\mu \phi \equiv \partial \phi / \partial x^\mu$, i.e.

$$A_\mu = \frac{\partial x'^\nu}{\partial x^\mu} A'_\nu. \tag{1.48}$$

Similarly, a rank (2,0) tensor $T^{\mu\nu}$ is defined as an object that transforms as the tensor product of two contravariant four-vectors, i.e.

$$A^{\mu\nu} = \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\sigma} A'^{\rho\sigma}. \tag{1.49}$$

By now, the generalization to arbitrary rank tensors should be obvious. As before, we can also contract a contravariant 4-vector and a covariant 4-vector to form a scalar which is invariant under coordinate transformations

$$A_\mu B^\mu = \frac{\partial x'^\alpha}{\partial x^\mu} A'_\alpha \frac{\partial x^\mu}{\partial x'^\beta} B'^\beta = A'_\alpha B'^\alpha \tag{1.50}$$

where in the last step we used

$$\frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x^\mu}{\partial x'^\beta} = \frac{\partial x'^\alpha}{\partial x'^\beta} \equiv \delta^\alpha{}_\beta. \tag{1.51}$$

1.2.2 Metric in arbitrary coordinate systems

In a Euclidean space, the spacetime interval can be written as

$$ds^2 = \eta_{ij} dx^i dx^j. \tag{1.52}$$

The same spacetime interval can be written in terms of the coordinates x'^μ in an arbitrary coordinate system as

$$ds^2 = \underbrace{\eta_{ij}\frac{\partial x^i}{\partial x'^\alpha}\frac{\partial x^j}{\partial x'^\beta}}_{g_{\alpha\beta}} dx'^\alpha dx'^\beta. \tag{1.53}$$

Therefore, the metric of an arbitrary system of coordinates $g_{\alpha\beta}$ is related to the Euclidean metric η_{ij} as

$$\boxed{g_{\alpha\beta} = \eta_{ij}\frac{\partial x^i}{\partial x'^\alpha}\frac{\partial x^j}{\partial x'^\beta}.} \tag{1.54}$$

This gives us a prescription for determining the metric in an arbitrary coordinate system, as demonstrated in the following example.

Example 1.2. Let us determine the metric in 3D spherical coordinates. Starting from the Euclidean metric

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^3, \tag{1.55}$$

one obtains the metric in spherical coordinates by making the following change of variables:

$$\begin{cases} x_1 &= r\sin\theta\cos\phi \\ x_2 &= r\sin\theta\sin\phi \\ x_3 &= r\cos\theta \end{cases} \tag{1.56}$$

which yields

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{1.57}$$

Alternatively, we can arrive at the same result using (1.54). We find that the non-zero components of the metric are

$$\begin{aligned} g_{rr} &= \eta_{ij}\frac{\partial x^i}{\partial r}\frac{\partial x^j}{\partial r} = 1 \\ g_{\theta\theta} &= \eta_{ij}\frac{\partial x^i}{\partial \theta}\frac{\partial x^j}{\partial \theta} = r^2\sin^2\theta + r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi = r^2 \\ g_{\phi\phi} &= \eta_{ij}\frac{\partial x^i}{\partial \phi}\frac{\partial x^j}{\partial \phi} = r^2\sin^2\theta\sin^2\phi + r^2\sin^2\sin^2\phi = r^2\sin^2\theta, \end{aligned}$$

in agreement with what we found earlier.

1.2.3 Volume element in arbitrary coordinate systems

When we go from Cartesian coordinates x^i to arbitrary coordinates x'^μ, the volume element is multiplied by the Jacobian of the transformation

$$dx'dy'dz' = \left(\det \frac{\partial x'^\mu}{\partial x^i}\right) dxdydz. \tag{1.58}$$

Here $\mu, i = 1, 2, 3$. The Jacobian can be obtained by taking the determinant of (1.54)

$$g = \left(\det \frac{\partial x^i}{\partial x'^\mu}\right)^2$$
$$\det \frac{\partial x'^\mu}{\partial x^i} = \frac{1}{\sqrt{g}} \tag{1.59}$$

where $g = \det g_{\mu\nu}$ and we have used the fact that the determinant of $\eta_{ij} = \mathrm{diag}(1,1,1)$ is 1. Thus

$$\sqrt{g}dx'dy'dz' = dxdydz \tag{1.60}$$

and the invariant volume form is given by

$$dV = \sqrt{g}dx'dy'dz' = dxdydz. \tag{1.61}$$

When we add time to the picture, we have a similar expression, but with $\sqrt{-g}$ instead of $\sqrt{g}$:

$$\sqrt{-g}dt'dx'dy'dz' = dtdxdydz. \tag{1.62}$$

Example 1.3. Volume element in spherical and cylindrical coordinates.
The metric in 3d spherical coordinates is

$$d\ell^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{1.63}$$

which translates to

$$g_{\mu\nu} = \mathrm{diag}\left(1, r^2, r^2\sin^2\theta\right) \tag{1.64}$$

whose determinant is

$$g = r^4 \sin^2\theta. \tag{1.65}$$

Hence, the volume element in spherical coordinates is

$$dV = r^2 \sin\theta dr d\theta d\phi. \tag{1.66}$$

On the other hand, the metric in 3d cylindrical coordinates is

$$d\ell^2 = dz^2 + d\rho^2 + \rho^2 d\phi^2 \tag{1.67}$$

which translates to

$$g_{\mu\nu} = \mathrm{diag}\left(1, 1, \rho^2\right) \tag{1.68}$$

whose determinant is

$$g = \rho^2. \tag{1.69}$$

Hence, the volume element in cylindrical coordinates is

$$dV = \rho dz d\rho d\phi. \tag{1.70}$$

1.2.4 Basis vectors, Christoffel symbols, and covariant derivatives

Let us consider for simplicity a 3D Cartesian coordinate system. We denote basis vectors in this system as $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$, or simply $\mathbf{e}_i$ with $i = x, y, z$. These basis vectors satisfy

$$\mathbf{e}_i \cdot \mathbf{e}_j = \eta_{ij}. \tag{1.71}$$

An arbitrary vector $\mathbf{V}$ can be projected on the Cartesian vector basis as

$$\mathbf{V} = V^i \mathbf{e}_i. \tag{1.72}$$

Let $x^\alpha = \xi, \eta, \rho$ be the coordinates of an arbitrary coordinate system, whose basis vectors are $\mathbf{e}'_\mu = \mathbf{e}'_\xi, \mathbf{e}'_\eta, \mathbf{e}'_\rho$. The same vector $\mathbf{V}$ above can be projected into the basis vectors of the arbitrary coordinate system as

$$\mathbf{V} = V'^\mu \mathbf{e}'_\mu. \tag{1.73}$$

Equating the above with (1.72) and recalling that a vector transforms as $V^i = V'^\mu \partial x^i / \partial x'^\mu$, we find that the basis vectors transform as

$$\mathbf{e}'_\mu = \frac{\partial x^i}{\partial x'^\mu} \mathbf{e}_i \tag{1.74}$$

i.e. like a covariant vector, where it is understood that there is a one-to-one mapping between the coordinates in the arbitrary system and the Cartesian coordinates

$$\begin{cases} x &= x(\xi, \eta, \rho) \\ y &= y(\xi, \eta, \rho) \\ z &= z(\xi, \eta, \rho). \end{cases} \tag{1.75}$$

The basis vectors provide us with an alternative way of defining the metric tensor

$$\boxed{g_{\alpha\beta} \equiv (\mathbf{e}_\alpha \cdot \mathbf{e}_\beta).} \tag{1.76}$$

Example 1.4. Using (1.74), we can relate the basis vectors in spherical coordinates to those in Cartesian coordinates

$$\begin{aligned} \mathbf{e}_r &= \frac{\partial x}{\partial r} \mathbf{e}_x + \frac{\partial y}{\partial r} \mathbf{e}_y + \frac{\partial z}{\partial r} \mathbf{e}_z = \sin\theta \cos\varphi \mathbf{e}_x + \sin\theta \sin\varphi \mathbf{e}_y + \cos\theta \mathbf{e}_z \\ \mathbf{e}_\theta &= \frac{\partial x}{\partial \theta} \mathbf{e}_x + \frac{\partial y}{\partial \theta} \mathbf{e}_y + \frac{\partial z}{\partial \theta} \mathbf{e}_z = \cos\theta \cos\varphi \mathbf{e}_x + \cos\theta \sin\varphi \mathbf{e}_y - \sin\theta \mathbf{e}_z \\ \mathbf{e}_\theta &= \frac{\partial x}{\partial \phi} \mathbf{e}_x + \frac{\partial y}{\partial \phi} \mathbf{e}_y + \frac{\partial z}{\partial \phi} \mathbf{e}_z = -\sin\varphi \mathbf{e}_x + \cos\varphi \mathbf{e}_y. \end{aligned} \tag{1.77}$$

We know that if V^i are the components of a vector $\mathbf{V} = V^i \mathbf{e}_i$ in Cartesian coordinates, then the derivative $\partial V^i / \partial x^j$ is a tensor. While this is true in Cartesian coordinates, the equivalent expression in arbitrary coordinates $\partial V^\mu / \partial x^\nu$ is not a tensor. Indeed, it transforms as

$$\begin{aligned} \frac{\partial V'^\mu}{\partial x'^\nu} &= \frac{\partial x^\rho}{\partial x'^\nu} \frac{\partial}{\partial x^\rho} \left(\frac{\partial x'^\mu}{\partial x^\sigma} V^\sigma \right) \\ &= \frac{\partial x^\rho}{\partial x'^\nu} \frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial V^\sigma}{\partial x^\rho} + \underbrace{\frac{\partial x^\rho}{\partial x'^\nu} \frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\sigma} V^\sigma}_{\neq 0} \\ &\neq \frac{\partial x^\rho}{\partial x'^\nu} \frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial V^\sigma}{\partial x^\rho} \quad \Longrightarrow \quad \text{not a tensor} \end{aligned} \tag{1.78}$$

This does not come as a surprise. The problem arises because we are not taking the derivative of a vector properly. Taking the derivative of a vector amounts to taking the difference between the vector at two neighbouring points, dividing it by a parameter representing the separation of the two points, and taking the limit in which this parameter goes to zero. What we have done above was comparing only the components of a vector at neighboring points, which is not the full story. The full story must account not only for the changes in the components of a vector but also for the changes in the basis vectors of the coordinate system. To do it properly, we start by writing a vector $\mathbf{V}$ in terms of its components in an arbitrary coordinate system

$$\mathbf{V} = V^\alpha \mathbf{e}_\alpha. \tag{1.79}$$

Then we take its derivative:

$$\frac{\partial \mathbf{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\mathbf{e}_\alpha + V^\alpha \frac{\partial \mathbf{e}_\alpha}{\partial x^\beta} \tag{1.80}$$

Note that $\partial \mathbf{V}/\partial x^\beta$ is a vector because the difference between two vectors is a vector. Since $\mathbf{e}_\alpha$ span the vector space, we can write

$$\frac{\partial \mathbf{e}_\alpha}{\partial x^\beta} = \Gamma^\gamma_{\alpha\beta}\mathbf{e}_\gamma, \tag{1.81}$$

where the coefficients $\Gamma^\gamma_{\alpha\beta}$ are known as Christoffel symbols. Thus, (1.80) becomes

$$\frac{\partial \mathbf{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\mathbf{e}_\alpha + V^\alpha \Gamma^\gamma_{\alpha\beta}\mathbf{e}_\gamma = \left(\frac{\partial V^\gamma}{\partial x^\beta} + V^\alpha \Gamma^\gamma_{\alpha\beta}\right)\mathbf{e}_\gamma, \tag{1.82}$$

where in the last step we simply relabeled the α's in the first term as γ's. In terms of its components, the above equation reads

$$\boxed{\nabla_\beta V^\gamma = \frac{\partial V^\gamma}{\partial x^\beta} + V^\alpha \Gamma^\gamma_{\alpha\beta}.} \tag{1.83}$$

Now $\nabla_\beta V^\gamma$, called *covariant derivative*, is a tensor. The covariant derivative can also be written in a shorthand notation as $V^\alpha_{;\beta}$. In the same spirit, the normal partial derivative can be written as $V^\alpha_{,\beta}$. It is also useful to define the *covariant differential* as

$$DV^\alpha = V^\alpha{}_{;\beta}\, dx^\beta. \tag{1.84}$$

Before going any further, let us try to understand what we have done in (1.82), physically. First, suppose that the arbitrary coordinate system we are using is a Cartesian one. What we did in (1.82) was the following. We identify two vectors, $\mathbf{V}(\mathbf{x_0})$ and $\mathbf{V}(\mathbf{x_0} + d\mathbf{x})$, and we want to take their difference. To do so, we *parallel transport* $\mathbf{V}(\mathbf{x_0} + d\mathbf{x})$ from its original position $\mathbf{x_0} + d\mathbf{x}$ to $\mathbf{x_0}$, that is, we move the vector while maintaining its direction parallel to itself throughout the process[2]. Once $\mathbf{V}(\mathbf{x_0} + d\mathbf{x})$ is successfully transported to $\mathbf{x_0}$, we take the difference between the transported $\mathbf{V}(\mathbf{x_0} + d\mathbf{x})$ with the $\mathbf{V}(\mathbf{x_0})$ that is residing there. This difference is the covariant difference $D\mathbf{V}$. As mentioned above, in an arbitrary frame, the basis vector changes. To stay parallel to itself, it needs to evolve accordingly to these changes. The aforementioned procedure can be summarized mathematically as follows

$$\begin{aligned} D\mathbf{V} &= V^\alpha(\mathbf{x} + d\mathbf{x})\mathbf{e}_\alpha(\mathbf{x} + d\mathbf{x}) - V^\alpha(\mathbf{x})\mathbf{e}_\alpha(\mathbf{x}) \\ &= \left(\frac{\partial V^\alpha}{\partial x^\beta} + V^\gamma \Gamma^\alpha_{\beta\gamma}\right)\mathbf{e}_\alpha dx^\beta. \end{aligned} \tag{1.85}$$

Remark 1.1. The relation (1.81) can be understood as a generalization of the Poisson formula

$$\frac{d\mathbf{e}_i}{dt} = \boldsymbol{\omega} \wedge \mathbf{e}_i, \tag{1.86}$$

which you have presumably seen in classical mechanics. It describes how the basis vectors of a coordinate system that is rotating with angular velocity $\boldsymbol{\omega}$ change with time.

[2] As mentioned previously, in an arbitrary coordinate system, the basis vectors change from place to place and so to stay parallel to itself means to evolve accordingly to these changes.

Remark 1.2. Here we show that the identity relating a covariant vector V_α and its contravariant counterpart V^β

$$V_\alpha = g_{\alpha\beta} V^\beta \tag{1.87}$$

is preserved under a coordinate transformation. Recall that V^β and $g_{\alpha\beta}$ transform as

$$V^\beta = \frac{\partial x^\beta}{\partial x'^\rho} V'^\rho \tag{1.88}$$

$$g_{\alpha\beta} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} g'_{\mu\nu}. \tag{1.89}$$

Plugging these into (1.87), we get

$$\begin{aligned} V_\alpha &= g_{\alpha\beta} V^\beta \\ &= \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} g'_{\mu\nu} \frac{\partial x^\beta}{\partial x'^\rho} V'^\rho \\ &= \frac{\partial x'^\mu}{\partial x^\alpha} \underbrace{\frac{\partial x'^\nu}{\partial x'^\rho}}_{=\delta^\nu{}_\rho} g'_{\mu\nu} V'^\rho \\ &= \frac{\partial x'^\mu}{\partial x^\alpha} g'_{\mu\rho} V'^\rho \\ &= \frac{\partial x'^\mu}{\partial x^\alpha} V'_\mu, \end{aligned} \tag{1.90}$$

which is how a covariant vector is expected to transform. Therefore, the relation (1.87) is preserved under a coordinate transformation. One can check that the same is true for the inverse relation

$$V^\beta = g^{\beta\alpha} V_\alpha. \tag{1.91}$$

Remark 1.3. The covariant derivative of a scalar field is equal to its partial derivative

$$\varphi_{;\alpha} = \varphi_{,\alpha} = \frac{\partial \varphi}{\partial x^\alpha}. \tag{1.92}$$

This is simply because the value of a scalar field does not depend on the coordinate system we are using. As a direct consequense, the derivatives of the basis vectors, which are the source of the difference between covariant and partial derivatives, do not enter the picture.

From the covariant derivative of a contravariant vector (1.83), we can derive the covariant derivative of a covariant vector. To that end, we start by taking the covariant differential of the contraction $V_\alpha U^\alpha$, making use of the last remark

$$\begin{aligned} D(\underbrace{V_\alpha U^\alpha}_{\text{scalar}}) &= (DV_\alpha) U^\alpha + V_\alpha (DU^\alpha) \\ \left(\frac{\partial V_\alpha}{\partial x^\beta} U^\alpha + \frac{\partial U^\alpha}{\partial x^\beta} V^\alpha \right) dx^\beta &= (DV_\alpha) U^\alpha + V_\alpha \left(\frac{\partial U^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\gamma\beta} U^\gamma \right) dx^\beta \end{aligned} \tag{1.93}$$

from which we can read off that

$$DV_\alpha U^\alpha = \left(\frac{\partial V_\alpha}{\partial x^\beta} - V_\chi \Gamma^\chi_{\alpha\beta} \right) U^\alpha dx^\beta \tag{1.94}$$

and using (1.84) we find the expression for the covariant derivative of a covariant vector

$$\boxed{V_{\alpha;\beta} = \frac{\partial V_\alpha}{\partial x^\beta} - \Gamma^\chi_{\alpha\beta} V_\chi.} \tag{1.95}$$

1.2.5 Christoffel symbols and metric tensor

Remark 1.4. Since both $V^{\alpha}_{;\beta}$ and $V^{\alpha;\beta}$ are tensors, the following relation

$$V_{\alpha;\beta} = g_{\alpha\mu} V^{\mu}{}_{;\beta} \tag{1.96}$$

holds in any coordinate system because we know that it is true in Cartesian coordinates. Using this relation, we can derive an important fact: the covariant derivative of the metric vanishes. The derivation goes as

$$\begin{aligned} \nabla_\beta V_\alpha &= g_{\alpha\mu} \nabla_\beta V^\mu \\ \nabla_\beta \left(g_{\alpha\mu} V^\mu \right) &= g_{\alpha\mu} \nabla_\beta V^\mu \\ V^\mu \nabla_\beta g_{\alpha\mu} + g_{\alpha\mu} \nabla_\beta V^\mu &= g_{\alpha\mu} \nabla_\beta V^\mu . \end{aligned} \tag{1.97}$$

It follows that

$$\boxed{\nabla_\beta g_{\alpha\mu} = 0.} \tag{1.98}$$

Up to this point, the covariant derivative of a rank (0,2) tensor is not yet known. We can deduce it by first contracting the tensor with a contravariant vector to get a covariant vector, whose covariant derivative we know how to evaluate. It can be shown that

$$T_{\alpha\beta;\mu} = \frac{\partial T_{\alpha\beta}}{\partial x^\beta} - \Gamma^\kappa_{\alpha\mu} g_{\kappa\beta} - \Gamma^\kappa_{\beta\mu} g_{\kappa\alpha}. \tag{1.99}$$

Combining this result with (1.98), we get the following equations

$$\begin{aligned} g_{\alpha\mu;\beta} &= \frac{\partial g_{\alpha\mu}}{\partial x^\beta} - \Gamma^\chi_{\alpha\beta} g_{\mu\chi} - \Gamma^\chi_{\mu\beta} g_{\alpha\chi} = 0 \\ g_{\alpha\beta;\mu} &= \frac{\partial g_{\alpha\beta}}{\partial x^\mu} - \Gamma^\chi_{\alpha\mu} g_{\chi\beta} - \Gamma^\chi_{\beta\mu} g_{\chi\alpha} = 0 \\ g_{\beta\mu;\alpha} &= \frac{\partial g_{\beta\mu}}{\partial x^\alpha} - \Gamma^\chi_{\beta\alpha} g_{\mu\chi} - \Gamma^\chi_{\mu\alpha} g_{\chi\beta} = 0. \end{aligned} \tag{1.100}$$

To find the Christoffel symbols from here, we need an extra piece of information relating $\Gamma^\gamma_{\alpha\beta}$ with $\Gamma^\gamma_{\beta\alpha}$, which we are going to derive in the following remark.

Remark 1.5. Since partial derivatives commute, the following holds in Cartesian coordinates

$$\varphi_{,i,j} = \varphi_{,i,j}. \tag{1.101}$$

Without leaving the Cartesian coordinates, the above identity can be written in a tensor form as

$$\varphi_{;\alpha;\beta} = \varphi_{;\beta;\alpha} \tag{1.102}$$

Now that it is written in a tensor form, the identity is guaranteed to hold in any arbitrary coordinates[3] Next we simplify the relation by citing the fact that covariant derivatives reduce to partial derivatives when applied to a scalar field

$$\varphi_{,\alpha;\beta} = \varphi_{,\beta;\alpha}. \tag{1.103}$$

Writing the covariant derivatives explicitly brings us to

$$\varphi_{,\alpha,\beta} - \Gamma^\kappa_{\alpha\beta} \varphi_\kappa = \varphi_{,\beta,\alpha} - \Gamma^\kappa_{\beta\alpha} \varphi_{,\kappa}. \tag{1.104}$$

Since the above equation is valid for any φ, we obtain the following identity

$$\boxed{\Gamma^\chi_{\alpha\beta} = \Gamma^\chi_{\beta\alpha}.} \tag{1.105}$$

[3] Rewriting an equation that is known to hold in a specific frame in a tensor form, thus making it valid in any coordinates, is a very useful trick in General Relativity. We will see it being used repeatedly.

Adding the first two equations in (1.100) and subtracting the result from the last one, we get

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = \underbrace{(\Gamma^{\chi}_{\alpha\beta} - \Gamma^{\chi}_{\beta\alpha})}_{=0} g_{\mu\chi} \underbrace{(\Gamma^{\chi}_{\mu\alpha} - \Gamma^{\chi}_{\mu\alpha})}_{=0} g_{\chi\beta} + \underbrace{(\Gamma^{\chi}_{\mu\beta} + \Gamma^{\chi}_{\beta\mu})}_{=2\Gamma^{\chi}_{\mu\beta}} g_{\chi\alpha}$$

where we have used (1.105). Hence, we can express the Christoffel symbols in terms of the metric as

$$\boxed{\Gamma^{\chi}_{\mu\beta} = \frac{1}{2} g^{\chi\alpha} \left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} \right).} \tag{1.106}$$

1.2.6 Christoffel symbols and coordinate change

The aim in this section is to derive an alternative definition of the Christoffel symbols in terms of basis vectors and in terms of the relations connecting the coordinate system in use to the Cartesian coordinate system. First, we take the scalar product between a basis vector $\mathbf{e}_\delta$ with (1.81)

$$\begin{aligned} \left(\mathbf{e}_\delta, \frac{\partial \mathbf{e}_\alpha}{\partial x^\beta} \right) &= \Gamma^{\gamma}_{\alpha\beta} \left(\mathbf{e}_\delta, \mathbf{e}_\gamma \right) \\ &= \Gamma^{\gamma}_{\alpha\beta} g_{\delta\gamma}, \end{aligned} \tag{1.107}$$

where we have used the definition of the metric in terms of the basis vectors (1.76). Next, we multiply both sides by $g^{\delta\rho}$

$$g^{\delta\rho} \left(\mathbf{e}_\delta, \frac{\partial \mathbf{e}_\alpha}{\partial x^\beta} \right) = \Gamma^{\gamma}_{\alpha\beta} g_{\gamma\delta} g^{\delta\rho}, \tag{1.108}$$

This brings us to yet another definition of Christoffel symbols in terms of basis vectors

$$\boxed{\Gamma^{\rho}_{\alpha\beta} = g^{\delta\rho} \left(\mathbf{e}_\delta, \frac{\partial \mathbf{e}_\alpha}{\partial x^\beta} \right),} \tag{1.109}$$

We can go further to express the right hand side in terms of coordinates only. Using the relation between the basis vectors in arbitrary and Cartesian coordinates (1.74), which also allows us to write

$$\frac{\partial \mathbf{e}_\alpha}{\partial x^\beta} = \frac{\partial}{\partial x^\beta} \left(\frac{\partial x^j}{\partial x^\alpha} \mathbf{e}_j \right) = \frac{\partial^2 x^j}{\partial x^\alpha \partial x^\beta} \mathbf{e}_j, \tag{1.110}$$

we can write

$$\left(\mathbf{e}_\delta, \frac{\partial \mathbf{e}_\alpha}{\partial x^\beta} \right) = \frac{\partial x^i}{\partial x^\delta} \frac{\partial^2 x^j}{\partial x^\alpha \partial x^\beta} \left(\mathbf{e}_i, \mathbf{e}_j \right) = \eta_{ij} \frac{\partial x^i}{\partial x^\delta} \frac{\partial^2 x^j}{\partial x^\alpha \partial x^\beta} \tag{1.111}$$

and so the Christoffel symbols become

$$\boxed{\Gamma^{\rho}_{\alpha\beta} = g^{\delta\rho} \eta_{ij} \frac{\partial x^i}{\partial x^\delta} \frac{\partial^2 x^j}{\partial x^\alpha \partial x^\beta}.} \tag{1.112}$$

In this form, the symmetry property of the Christoffel symbols under the interchange of the lower two indices, i.e. α and β, is manifest. By rewriting the metric in arbitrary coordinates $g^{\rho\sigma}$ in terms of the Cartesian metric, we can write it in a simpler form

$$\begin{aligned} \Gamma^{\rho}_{\alpha\beta} &= \left(\frac{\partial x^\rho}{\partial x^k} \frac{\partial x^\delta}{\partial x^\ell} \eta^{k\ell} \right) \eta_{ij} \frac{\partial x^i}{\partial x^\delta} \frac{\partial^2 x^j}{\partial x^\alpha \partial x^\beta} \\ &= \frac{\partial x^\rho}{\partial x^k} \underbrace{\delta^i_\ell \eta^{k\ell} \eta_{ij}}_{\delta^k_j} \frac{\partial^2 x^j}{\partial x^\alpha \partial x^\beta} \\ &= \frac{\partial x^\rho}{\partial x^j} \frac{\partial^2 x^j}{\partial x^\alpha \partial x^\beta} \end{aligned} \tag{1.113}$$

where in going from the first to second line we combined $\partial x^\delta/\partial x^\ell$ and $\partial x^i/\partial x^\delta$ to form a δ^i_ℓ.

1.2.7 Newtonian equation of motion in arbitrary coordinates systems

Remark 1.6. Levi-Civita tensor.
We have previously defined $\epsilon_{ijk\ell}$ in Euclidean space as a completely antisymmetric set of numbers with the convention $\epsilon_{1234} = 1$. The exact same set of numbers $\epsilon_{\alpha\beta\gamma\delta}$ can also be defined in arbitrary coordinates. As $\epsilon_{ijk\ell}$ and $\epsilon_{\alpha\beta\gamma\delta}$, so defined, have the same values in any coordinate system, they are not tensors. For this reason, $\epsilon_{ijk\ell}$ (and $\epsilon_{\alpha\beta\gamma\delta}$) are called Levi-Civita *symbols* (instead of tensors). The question we are going to address now is: how do we turn the Levi-Civita symbol into a tensor? Let us define the Levi-Civita tensor $E^{\alpha\beta\gamma\delta}$ as a tensor whose values in Cartesian coordinates match with those of the Levi-Civita symbol $\epsilon_{ijk\ell}$. Hence, we have the relation

$$E^{\alpha\beta\gamma\delta} = \frac{\partial x^\alpha}{\partial x'^i}\frac{\partial x^\beta}{\partial x'^j}\frac{\partial x^\gamma}{\partial x'^k}\frac{\partial x^\delta}{\partial x'^l}\epsilon^{ijkl}. \tag{1.114}$$

Since, collectively, the derivative factors are completely symmetric with respect to changes of the type $\alpha \to \beta$ and $i \to j$ (both carried out at once) and ϵ^{ijkl} is completely antisymmetric, $E^{\alpha\beta\gamma\delta}$ must also be completely antisymmetric, which means that $E^{\alpha\beta\gamma\delta}$ is proportional to $\epsilon^{\alpha\beta\gamma\delta}$:

$$E^{\alpha\beta\gamma\delta} = C\epsilon^{\alpha\beta\gamma\delta} \tag{1.115}$$

To determine C, let us multiply both sides by $\epsilon_{\alpha\beta\gamma\delta}$

$$\begin{aligned} E^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} &= C\,\underbrace{\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta}}_{=4!} \\ \frac{\partial x^\alpha}{\partial x'^i}\frac{\partial x^\beta}{\partial x'^j}\frac{\partial x^\gamma}{\partial x'^k}\frac{\partial x^\delta}{\partial x'^l}\epsilon^{ijkl}\epsilon_{\alpha\beta\gamma\delta} &= 4!C \\ 4!\det\left(\frac{\partial x^\alpha}{\partial x'^j}\right) &= 4!C, \\ \frac{1}{\sqrt{-g}} &= C \end{aligned} \tag{1.116}$$

where the identity

$$\det(A) = \frac{1}{n!}\epsilon_{i_1,\ldots,i_n}\epsilon_{j_1,\ldots,j_n}(a_{i_1,j_1}\cdots a_{i_n,j_n}). \tag{1.117}$$

and the 4D spacetime analog of (1.59), namely

$$\det\frac{\partial x^\alpha}{\partial x'^i} = \frac{1}{\sqrt{-g}} \tag{1.118}$$

were used. Thus

$$E^{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{-g}}\epsilon^{\alpha\beta\gamma\delta}. \tag{1.119}$$

If we start with the indices at the bottom and repeat the same steps we would get

$$E_{\alpha\beta\gamma\delta} = \sqrt{-g}\epsilon_{\alpha\beta\gamma\delta}. \tag{1.120}$$

Remark 1.7. Dimensionality of space.
Recall that the existence of a one-to-one relation between the coordinates in an arbitrary system and those in the Cartesian system requires the Jacobian of the transformation to be nonzero, $\det(\partial x^i/\partial x^\alpha) \neq 0$. This can be true only if the number of x^i coordinates n is the same as the number of x^α coordinates N. Let us check

what happens if the latter is not satisfied by supposing that $N > n$. In that case, the metric

$$g_{\alpha\beta} = \frac{\partial x^i}{\partial x^\alpha} \frac{\partial x^j}{\partial x^\beta} \eta_{ij} \tag{1.121}$$

could still be defined. The problem is that its determinant would vanish, $\det g_{\alpha\beta} = 0$, meaning that the volume form would be zero.

Now, we turn to the following question: suppose that you are given a metric $g_{\alpha\beta}$, how can you tell from it the dimensionality of the space it describes? For example, the following metric

$$d\ell^2 = dx^2 + dy^2 + 2dxdy \tag{1.122}$$

may appear as if it describes a 2 dimensional space. However, by defining $z = x + y$, we can rewrite it as

$$d\ell^2 = (dx + dy)^2 = dz^2 \tag{1.123}$$

which clearly describes a 1 dimensional space. In general, the dimension of the space described by a metric $g_{\alpha\beta}$ is given by the rank of $g_{\alpha\beta}$.

Now that we have studied various aspects of the connection between Cartesian and arbitrary coordinates, we are ready to rewrite the Newtonian equation of motion in arbitrary coordinates. As a starting point, we write the Newtonian equation of motion for a free massive particle in Cartesian coordinates

$$\frac{d^2 x^i}{ds^2} = 0, \tag{1.124}$$

with $ds^2 = \eta_{ij} dx^i dx^j$. Using the relation (1.43), we rewrite the above as

$$\frac{d}{ds}\left(\frac{\partial dx^i}{\partial x^\mu}\frac{dx^\mu}{ds}\right) = \frac{\partial x^i}{\partial x^\mu}\frac{d^2 x^\mu}{ds^2} + \frac{\partial^2 x^i}{\partial x^\nu \partial x^\mu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0. \tag{1.125}$$

Next, we multiply it with $\partial x^\lambda / \partial x^i$

$$\frac{\partial x^\lambda}{\partial x^i}\frac{\partial x^i}{\partial x^\mu}\frac{d^2 x^\mu}{ds^2} + \frac{\partial x^\lambda}{\partial x^i}\frac{\partial^2 x^i}{\partial x^\nu \partial x^\mu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0. \tag{1.126}$$

Noting that $(\partial x^\lambda/\partial x^i)(\partial x^i/\partial x^\mu) = \delta^\lambda_\mu$, the above equation simplifies to

$$\frac{d^2 x^\lambda}{ds^2} + \underbrace{\frac{\partial x^\lambda}{\partial x^i}\frac{\partial^2 x^i}{\partial x^\nu \partial x^\mu}}_{\Gamma^\lambda_{\mu\nu}}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{1.127}$$

and the equation of motion for a free massive particle in arbitrary coordinates reads

$$\frac{d^2 x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0. \tag{1.128}$$

The above equation, however, does not apply for a massless particle because in that case $ds = 0$. To obtain an analogous equation for a massless particle, we simply replace the ds's with the differential of time $d\sigma$ in a local Cartesian coordinate system:

$$\frac{d^2 x^\lambda}{d\sigma^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} = 0. \tag{1.129}$$

It is quite straightforward to include the electromagnetic field in (1.128). If instead of (1.124) we had started with

$$\frac{d^2 x^i}{ds^2} = qF^{ij}\frac{dx_j}{ds} \tag{1.130}$$

we would have arrived at

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = qF^\lambda_\alpha\frac{dx^\alpha}{ds}, \tag{1.131}$$

which should be complemented with the Maxwell equations to complete the picture. The tensor forms of the Maxwell equations are obtained by the following replacements

$$\begin{aligned} F^{\mu\nu}{}_{,\nu} = j^\nu \quad &\longrightarrow \quad F^{\mu\nu}{}_{;\nu} = j^\nu \\ \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma,\nu} = 0 \quad &\longrightarrow \quad E_{\mu\nu\rho\sigma}F^{\rho\sigma;\nu} = 0. \end{aligned} \tag{1.132}$$

1.2.8 Summary of mathematics

One-to-one relations $x^i = x^i(x^\alpha)$ between Cartesian coordinates x^i and arbitrary coordinates x^α exist if the Jacobian of the transformation is nonzero:

$$\det\frac{\partial x^i}{\partial x^\alpha} \neq 0.$$

The length of a space interval is coordinate independent and is given by

$$d\ell^2 = \eta_{ij}\,dx^i dx^j = g_{\alpha\beta}\,dx^\alpha dx^\beta,$$

with

$$\eta_{ij} = \underbrace{\mathrm{diag}(1,1,1)}_{\text{Euclidean}} \quad \text{or} \quad \underbrace{\mathrm{diag}(1,-1,-1,-1)}_{\text{Minkowskian}}$$

If we know how to relate two sets of arbitrary coordinates to the Cartesian coordinates, $x^i = x^i(x'^\alpha)$ and $x_i = \tilde{x}^i(x^\beta)$, then we can directly relate the two sets of arbitrary coordinates

$$x^\alpha = x'^\alpha(x^\beta)$$

and their differentials

$$dx'^\alpha = \frac{dx'^\alpha}{dx^\beta}dx^\beta$$

and their derivatives

$$\frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial}{\partial x^\beta}.$$

The last two relations allow us to define contravariant and covariant vectors. Contravariant and covariant vectors are objects that transform as dx^α and $\partial/\partial x^\alpha$ respectively

$$\begin{aligned} V^\alpha \to V'^\alpha &= \frac{\partial x'^\alpha}{\partial x^\beta}V^\beta \quad \text{contravariant} \\ V_\alpha \to V'_\alpha &= \frac{\partial x^\beta}{\partial x'^\alpha}V_\beta \quad \text{covariant.} \end{aligned}$$

Similar definitions apply for tensors, e.g.

$$V^{\alpha\beta} \to V'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial x'^\beta}{\partial x^\delta}V^{\gamma\delta}.$$

Contracting a covariant vector U_α and a contravariant vector V^α gives us a quantity that is invariant under coordinate transformations

$$U'_\alpha V'^\alpha = U_\alpha V^\alpha.$$

Basis vectors $\mathbf{e}_\alpha$ in arbitrary coordinates are related to those in Cartesian coordinates $\mathbf{e}_i$ as follows

$$\mathbf{e}_\alpha = \frac{\partial x^i}{\partial x^\alpha}\mathbf{e}_i.$$

In Cartesian coordinates the metric can be expressed in terms of basis vectors as

$$\eta_{ij} = (\mathbf{e}_i, \mathbf{e}_j)$$

and, similarly, in arbitrary coordinates we have

$$g_{\alpha\beta} = (\mathbf{e}_\alpha, \mathbf{e}_\beta).$$

The change in a vector field can be written as

$$\mathbf{V}(x + dx) - \mathbf{V} = d\mathbf{V} = V^\alpha_{;\mu} \mathbf{e}_\alpha dx^\mu,$$

where the covariant derivative $V^\alpha_{;\mu}$ is given by

$$V^\alpha_{;\mu} = \frac{\partial V^\alpha}{\partial x^\mu} + \Gamma^\alpha_{\beta\mu} V^\beta$$

which is a tensor. The Christoffel symbol $\Gamma^\alpha_{\beta\mu}$ is defined by the relation

$$\frac{\partial \mathbf{e}_\beta}{\partial x^\mu} = \Gamma^\alpha_{\beta\mu} \mathbf{e}_\alpha.$$

It obeys the symmetry property $\Gamma^\alpha_{\beta\mu} = \Gamma^\alpha_{\mu\beta}$ and can be expressed in terms of the metric or in terms of coordinate transformations.

The spacetime volume element in arbitrary coordinates is given by

$$dV = \sqrt{-g} dx^0 dx^1 dx^2 dx^3.$$

Any covariant equation in Cartesian coordinates can be rewritten as a covariant equation in arbitrary coordinates by replacing the ordinary partial derivatives by covariant derivatives. For example,

$$\begin{aligned} \partial_\mu J^\mu = 0 \quad &\rightarrow \quad J^\mu_{;\mu} = 0 \\ \partial_\mu F^{\mu\nu} = J^\nu \quad &\rightarrow \quad F^{\mu\nu}_{;\mu} = J^\nu. \end{aligned}$$

2 GENERAL RELATIVITY

2.1 THEORY OF GRAVITY BEFORE EINSTEIN

There were many attempts at describing the movements of planets in the solar system before the development of the Newtonian theory of gravity. Most of them do not even come close to describing reality as we know it today. Nevertheless, a few principles remain (more or less) accurate even today, namely the three *Kepler's laws*:

1. Every planet in the solar system moves in an elliptical orbit with the Sun located at one of its foci.
2. The orbit radius of a planet sweeps equal portions of the ellipse area in equal time intervals.
3. The squared period of a planet is proportional to the cube of the semi-major axis of its orbit: $T^2 = ca^3$.

Then, Newton came along and introduced his theory of gravity which, among other things, reproduces the three Kepler's laws. The theory states that two masses, m_1 and m_2, attract each other with a gravitational force that is inversely proportional to the squared distance r between them

$$\mathbf{F} = -G_N \frac{m_1 m_2}{r^3} \mathbf{r}. \tag{2.1}$$

Furthermore, the theory also states that the inertial mass (one that appears in the right hand side of $\mathbf{F} = m\mathbf{a}$) is equivalent to the gravitational mass (one that appears in $Gm_1 m_2/r^2$), i.e. $m_i \equiv m_g$.

2.2 GENERAL REQUIREMENTS FOR A RELATIVISTIC THEORY OF GRAVITY

At this point, we are of course tempted to find the relativistic version of the Newton's law of gravity. The first problem we are facing is that (2.1) is not manifestly Lorentz invariant. Incidentally, the Coulomb's law

$$F = -\frac{q_1 q_2}{r^3} \mathbf{r} \tag{2.2}$$

whose relativistic formulation we know well (the Lorentz force law plus Maxwell's equations), has a form very similar to (2.1). Let us see if we can extract a few hints from it. To go from the Coulomb's law to its relativistic formulation, we would need to: introduce the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ (to ensure all interactions are local), introduce non-trivial equations the $\mathbf{E}$ and $\mathbf{B}$ fields must satisfy (Maxwell's equations), and specify how the $\mathbf{E}$ and $\mathbf{B}$ fields exert forces on charges (Lorentz force law). In analogy with $\Box A^\mu = j^\mu$ of electrodynamics, we expect the relativistic formulation of gravity to include a field which obeys a differential equation of the form

$$\text{(some diff. operator)[field]} = \text{[source]} \tag{2.3}$$

and, in analogy with the force law $F^\mu = F^{\mu\nu} u_\nu$, a gravitational force law of the form

$$\text{[force]} = \text{[combinations of fields and 4-velocity of the particle]} \tag{2.4}$$

is expected. In electrodynamics, the fields are A_μ (4-vector) and $F_{\mu\nu}$ (antisymmetric tensor) and the sources are j^μ (4-vector). To sum up, a relativistic theory of gravity must have the following ingredients:

- Representation of gravitational field (Scalar? Vector? Tensor?)
- Representation of the source of gravitational field (Scalar? Vector? Tensor?)
- Covariant equation for the gravitational field.
- Covariant expression of the gravitational force.

2.3 FAILED ATTEMPTS AT CONSTRUCTING A RELATIVISTIC THEORY OF GRAVITY

2.3.1 Gravity as a 4-vector field

Since the Coulomb's law looks almost like the Newton's law of gravity, one would be inclined to introduce a 4-vector field G_μ to represent the gravitational field and consider the correspondence: $G_\mu \leftrightarrow A_\mu$ and $m \leftrightarrow q$. Bringing the analogy further, we expect the equations for G_μ to be the Maxwell's equations with A_μ replaced by G_μ and q replaced by m. However, without even writing the equations, we can see why this will not work. While a vector field may give both attractive and repulsive forces, the gravitational interaction between matter is always attractive. So, we conclude that representing the gravitational field as a 4-vector field does not work.

Remark 2.1. The gravitational interaction between antiparticles, e.g. positron and positron, is also attractive. This was the outcome of the analysis of KK^0 mixing experiments occurring in Earth's gravitational field.

2.3.2 Gravity as a scalar field

This time, instead of starting from the inverse square law, let us start with the Poisson's equation for gravity sourced by a point mass M

$$\nabla^2 \varphi = -\delta^3(\mathbf{x}) GM, \tag{2.5}$$

where φ is the gravitational potential. The equation motion of a test particle m in this gravitational potential is

$$m \frac{d^2 \mathbf{x}}{dt^2} = -m \nabla \varphi. \tag{2.6}$$

One possible relativistic generalization of (2.5) and (2.6) can be obtained by making the following substitutions

- $\mathbf{x} \to x^\mu$
- $\nabla^2 \longrightarrow \Box$
- $dt \longrightarrow ds = \sqrt{dx_\mu dx^\mu}$
- $\nabla \longrightarrow \nabla^\mu = \frac{\partial}{\partial x^\mu}$

With these substitutions, the equation of gravitational potential and the equation of motion become

$$-\Box \varphi = G\rho(\mathbf{x}) \tag{2.7}$$

$$m \frac{d^2 x^\mu}{ds^2} = m \nabla^\mu \varphi \tag{2.8}$$

where $\rho(\mathbf{x})$ is the mass density. In the limit of small time derivatives and speeds, these equations reduce to the non-relativistic equations we started with.

Unfortunately, this formulation of gravity is not consistent. Indeed, contracting both sides of (2.8) with $\frac{dx_\mu}{ds}$, we get

$$\begin{aligned} m\frac{dx_\mu}{ds}\frac{d^2x^\mu}{ds^2} &= m\frac{\partial\varphi}{\cancel{\partial x_\mu}}\frac{\cancel{dx_\mu}}{ds} \\ m\frac{d}{ds}\underbrace{\left(\frac{1}{2}\frac{dx^\mu}{ds}\frac{dx_\mu}{ds}\right)}_{=\frac{ds^2}{ds^2}=1} &= m\frac{d\varphi}{ds} \\ 0 &= \frac{d\phi}{ds}. \end{aligned} \tag{2.9}$$

This result tells us that, regardless of the physical system under consideration, the gravitational potential does not change along any trajectory of the test particle, which is absurd. One way to overcome this problem is by modifying the right hand side of (2.8) such that it automatically gives zero when contracted with $\frac{dx_\mu}{ds}$

$$m\frac{d^2x^\mu}{ds^2} = m\left(\nabla^\mu\varphi - \frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{\partial\varphi}{\partial x^\nu}\right). \tag{2.10}$$

Indeed,

$$\begin{aligned} m\left(\frac{\partial\varphi}{\partial x_\mu} - \frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{\partial\varphi}{\partial x^\nu}\right)\frac{dx_\mu}{ds} &= m\Bigg(\frac{d\varphi}{ds} - \underbrace{\frac{dx_\mu dx^\mu}{ds^2}}_{=\frac{ds^2}{ds^2}=1}\frac{\cancel{dx^\nu}}{ds}\frac{\partial\varphi}{\cancel{\partial x^\nu}}\Bigg) \\ &= m\left(\frac{d\varphi}{ds} - \frac{d\varphi}{ds}\right) = 0 \end{aligned}$$

The pair of equations (2.7) and (2.10) now form a consistent relativistic theory of gravity, at least theoretically.

The true test of scalar gravity is whether or not it agrees with experiment. The answer turns out to be no. As we will show below, scalar gravity predicts no light deflection (ie. modification of the light trajectory due to the presence of massive objects) while experiments show otherwise. Consider a massive but very light particle moving in a static gravitational field due to a massive body in Minkowskian coordinates. The $\mu = 0$ component of (2.10) reads

$$\begin{aligned} \frac{d}{ds}\left(\frac{dx^0}{ds}\right) &= \underbrace{\frac{\partial\varphi}{\partial t}}_{=0 \text{ for static potential}} - \frac{1}{\sqrt{1-v^2}}\frac{1}{\sqrt{1-v^2}}\underbrace{\frac{d\varphi}{dt}}_{\neq 0 \text{ along the trajectory}} \\ \frac{1}{\sqrt{1-v^2}}\frac{d}{dt}\left(\frac{1}{\sqrt{1-v^2}}\right) &= -\frac{1}{1-v^2}\frac{d\phi}{dt} \\ \frac{1}{\gamma}\frac{d}{dt}\gamma &= -\frac{d\varphi}{dt} \end{aligned} \tag{2.11}$$

where we have used $ds = \sqrt{1-v^2}dt$ and $\gamma \equiv 1/\sqrt{1-v^2}$. The $\mu = i$ component of (2.10) reads

$$\begin{aligned} \frac{1}{\sqrt{1-v^2}}\frac{d}{dt}\left(\frac{v_i}{\sqrt{1-v^2}}\right) &= \frac{\partial\varphi}{\partial x^i} - \frac{v_i}{\sqrt{1-v^2}}\frac{d\varphi}{ds} \\ \frac{1}{1-v^2}\frac{dv_i}{dt} + \frac{v_i}{\sqrt{1-v^2}}\frac{d}{dt}\left(\frac{1}{\sqrt{1-v^2}}\right) &= \frac{\partial\varphi}{\partial x^i} - \frac{v_i}{\sqrt{1-v^2}}\frac{d\varphi}{ds} \\ \frac{1}{1-v^2}\frac{dv_i}{dt} - \frac{v_i}{1-v^2}\frac{d\varphi}{dt} &= \frac{\partial\varphi}{\partial x^i} - \frac{v_i}{1-v^2}\frac{d\varphi}{dt} \\ \frac{dv_i}{dt} &= (1-v^2)\frac{\partial\phi}{\partial x_i} \end{aligned} \tag{2.12}$$

where from the second to third line we have used (2.11). Integrating (2.11) gives

$$\log\gamma + \varphi = C \tag{2.13}$$

or

$$\gamma = Ce^{-\varphi} \tag{2.14}$$

or, written in terms of velocity

$$v^2 = 1 - (1 - v_0^2)e^{2\varphi} \tag{2.15}$$

where we have picked the C that satisfies the boundary conditions that $\phi \to 0$ and $v \to v_0$ as $x \to \infty$. The above equation says that if $v_0 = 1$ then $v = 1$ at all time. According to (2.12), this implies that $dv_i/dt = 0$, meaning that scalar gravity predicts no light deflection, in contrary to what experiments tell us. Thus, scalar gravity cannot be the theory of gravity.

As we will see in the next section, Einstein's theory of gravity, our current best theory of gravity, is based on the equivalence principle (gravitational mass = inertial mass).

2.4 EQUIVALENCE PRINCIPLE

We call the mass m_i that appears on the right hand side of the Newton's law of motion $\mathbf{F} = m_i\mathbf{a}$ *inertial mass* and the mass m_g that appears in the formula for the force of gravity in a uniform gravitational field $\mathbf{F} = m_g\mathbf{g}$ *gravitational mass*. The equivalence principle states that the inertial and gravitational mass are the same. If the equivalence principle is respected, then the speed of free falling objects does not depend on their (inertial) masses. This was (roughly) demonstrated by Galileo in his famous Leaning Tower of Pisa experiment. However, it may still be possible that the two masses are nearly the same but not quite. If the two masses are different, then the acceleration of a free falling object is given by

$$\mathbf{a} = \frac{m_g}{m_i}\mathbf{g} \tag{2.16}$$

instead of simply $\mathbf{g}$. The *Eotvos experiment* in Hungary looked for possible deviations of m_i from m_g. In this experiment, we hang a dumbbell with different masses attached at its two ends. These masses experience two types of forces: Earth's gravitational pull and inertial force due to Earth's rotation. The dumbbell's arm lengths, ℓ_A and ℓ_B, are chosen such that the following is satisfied to a high precision

$$m_A^g \ell_A = m_B^g \ell_B. \tag{2.17}$$

This ensures that the horizontal torques caused by the vertical gravitational forces balance. On the other hand, the net vertical vertical torque due to the horizontal inertial forces is given by

$$\tau_V \propto (m_A^i \ell_A - m_B^i \ell_B). \tag{2.18}$$

If the inertial and gravitational mass are the same, that is, if

$$\frac{m_A^i}{m_A^g} = \frac{m_B^i}{m_B^g} \tag{2.19}$$

then (2.18) together with (2.17) imply that the vertical torque is zero $\tau_V = 0$. Therefore, by attempting to measure the possible rotation of the dumbbell due to nonzero τ_V with intricate optics, we can constrain the fractional difference between the inertial and gravitational mass. The current constraint is very stringent

$$\frac{|m_i - m_g|}{m_g} \lesssim 2 \times 10^{-13} \tag{2.20}$$

For this reason, we can rely on the equivalence principle as the basis of general relativity with a high confidence.

Remark 2.2. Scalar gravity with electromagnetism.
As an example of a theory of gravity that violates the equivalence principle, here we consider a modified scalar gravity with electromagnetism incorporated. Below, we modify (2.7) by adding the only Lorentz invariant terms we can construct out of $F_{\mu\nu}$

$$\Box\varphi = \underbrace{-G\rho(\mathbf{x}) + aF_{\mu\nu}F^{\mu\nu} + b\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}}_{\text{gravity source}}. \tag{2.21}$$

Recall that

$$F_{\mu\nu}F^{\mu\nu} \propto E^2 - B^2 \tag{2.22}$$

$$\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} \propto \mathbf{E}\cdot\mathbf{B} \tag{2.23}$$

$$\epsilon_{EM} = \frac{1}{2}\left(E^2 + B^2\right) \tag{2.24}$$

This means that different EM configurations which have the same electromagnetic energy ϵ_{em} will gravitate in different manners (e.g. take $|E| = |B| = 1$ and vary the relative orientation). We thus expect the equivalence principle to be violated in this theory.

In order to see an important consequence of the equivalence principle, consider the following configurations:

1. A particle in an inertial frame under the influence of a gravitational field $\mathbf{g}$.
2. A particle in a non-inertial frame that is moving with acceleration $\mathbf{g}$ in the absence of gravitational field.

In both cases, any particle, regardless of its mass, would follow the exact same trajectory if released with the same initial conditions (spacetime position and velocity). Therefore, as far as the motion of the particle is concerned, the two configurations are indistinguishable. This observation may serve as a starting point to state the equivalence principle in other terms: there is no means for an observer which observes a free particle moving with a constant acceleration to determine whether they are themselves accelerating or the particle is subject to a gravitational force.

Remark 2.3. Lift experiment.
The *Lift experiment* is a *gedänken experiment* (thought experiment) proposed by A. Einstein to illustrate an important consequence of the equivalence principle: that the effect of gravity can be removed locally by going to an appropriate frame of reference. This can only be done exactly at one point, or approximately in a small region around that point. Consider an observer confined in a free-falling box with no window. The coordinates ξ^α belonging to the coordinate system which is falling together with the box is related to those of the (inertial) laboratory coordinates x'^i as

$$\xi^\alpha = \xi^\alpha(x'^i). \tag{2.25}$$

If the equivalence principle is respected, there exists a coordinate system, namely the free-falling coordinate system, where the gravitational force on a particle is exactly canceled by the inertial force exerted on it, leaving the particle free of all forces. In that reference frame, the equation of motion of the particle is given by

$$\frac{d^2\xi^i}{dt^2} = 0 \tag{2.26}$$

with $i = 1, 2, 3$. Its relativistic generalization is

$$\frac{d^2\xi^\alpha}{ds^2} = 0 \tag{2.27}$$

with $\alpha = 0, 1, 2, 3$ and

$$ds^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta \tag{2.28}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric. Comparing (2.27) with the equation of motion in arbitrary coordinates

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \tag{2.29}$$

which we have derived in (1.128), we see that (2.27) resembles the equation of motion of a free particle in Minkowskian coordinates with $\Gamma^\lambda_{\mu\nu} = 0$ and $g_{\mu\nu} = \eta_{\mu\nu}$. Therefore, as far as the equation of motion of a particle is concerned, the free-falling coordinate system is effectively a Minkowskian coordinate system. Notice that the presence of gravitational/inertial force comes hand-in-hand with the departure from the Minkowski metric, which suggests that gravity could be encoded in the metric.

Remark 2.4 (**important**). In general, we cannot get an arbitrary metric field $\tilde{g}_{\mu\nu}(x)$ from a globally Minkowskian one $g_{\mu\nu}(x) = \eta_{\mu\nu}$ and vice versa via a single coordinate transformation.
We have just seen in the previous remark that, in the presence of an arbitrary gravitational field, we can always find a local free-falling coordinate system in which the effect of gravity is removed. In such a coordinate system, the effective metric inferred from the motion of a test particle is Minkowskian. The required coordinate transformation to cancel the effect of gravity is in general different from point to point and we would need infinitely many different coordinate transformations to completely remove the effect of gravity at all points. Similarly, one can prove by counting degrees of freedom that there is no single coordinate transformation that brings an arbitrary metric field $\tilde{g}(x)$ to a Minkowskian one $g_{\mu\nu}(x) = \eta_{\mu\nu}$, or vice versa. The best that one can hope to do is to set $g_{\mu\nu}(x_p) = \eta_{\mu\nu}$ and $\partial_\rho g_{\mu\nu}(x_p) = 0$ at a point x_p, but there is not enough freedom in making coordinate transformations to allow us to also set the higher derivatives to zero: $\partial_\rho\partial_\sigma g_{\mu\nu}(x_p) = 0$, $\partial_\rho\partial_\sigma\partial_\gamma g_{\mu\nu}(x_p) = 0$, etc which are required if we are to have a globally Minkowskian metric. This suggests that the presence of gravitational field can be tied with non-Minkowskian metric, giving us a further motivation to describe gravitational field in terms of the metric $g_{\mu\nu}$.

Remark 2.5. Examples of curved spaces.
To study a curved space, i.e. a space whose metric is not equivalent to the Minkowski metric, it is often helpful to embed the space in a higher-dimensional flat space, where things are better understood. For simplicity, let us take a two-dimensional curved space and embed it in a three-dimensional Euclidean space, in which the curved space appears as the surface defined by $F(x, y, z) = 0$. Particles constrained to this surface would "feel" a different metric from that of the Euclidean space. We call such a metric *induced metric*. Let us try to calculate the induced metric on an arbitrary two-dimensional surface. Taking the differential of the surface equation $F(x, y, z)$ gives

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz = 0. \tag{2.30}$$

Using it to eliminate the dz in the Euclidean metric

$$d\ell^2 = dx^2 + dy^2 + dz^2 \tag{2.31}$$

we get

$$d\ell^2 = dx^2 + dy^2 + \frac{1}{(\partial F/\partial z)^2} \left(\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy \right)^2_{z=z(x,y)} \tag{2.32}$$

which is the induced metric on the surface. As an example, consider a sphere defined by

$$x^2 + y^2 + z^2 = R^2. \tag{2.33}$$

The above constraint is automatically satisfied if we work with the following parameterization

$$
\begin{aligned}
z &= R\cos\theta \\
y &= R\sin\theta\cos\phi \\
x &= R\sin\theta\sin\phi.
\end{aligned}
$$

Plugging these into the Euclidean metric (2.31), we get the induced metric

$$d\ell^2 = R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.34}$$

with $0 \leqslant \theta \leqslant \pi$ and $0 < \phi \leqslant 2\pi$. We can bring the metric to the Euclidean form locally but not globally by a single coordinate transformation, e.g. for the points close to the line $\theta = \pi/2$. The then metric reduces to

$$d\ell^2 = R^2\left(d\theta^2 + d\phi^2\right) \tag{2.35}$$

which looks like the 2d Euclidean metric, apart from the R^2 factor, which can easily be rescaled away by a further coordinate transformation. In fact, for each point on the sphere we can find a coordinate system where the metric is Euclidean by shifting the coordinates θ and ϕ appropriately. However, we stress here that this is only doable for one point at a time.

2.5 GEODESIC EQUATION FROM THE LEAST ACTION PRINCIPLE

Recall that in classical mechanics a particle moves from one point, say A, to another point, say B, following a motion which minimizes the action

$$S = \int_0^T L(x, \dot{x})dt, \tag{2.36}$$

with the boundary conditions $x(0) = A$ and $x(T) = B$. Requiring that the action is stationary leads us to the *Euler-Lagrange* equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{2.37}$$

Our aim now is to construct the curved-spacetime version of the free-particle least action principle. Considering that the action must be a scalar, i.e. Lorentz invariant, arguably the simplest action we can write down is proportional to the spacetime interval $\int ds$. By convention, the proportionality constant is taken to be $-m$, so that the action is given by

$$S = -m\int ds \tag{2.38}$$

or

$$\boxed{S = -m\int d\tau\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)^{1/2}} \tag{2.39}$$

where τ can be any variable that parameterizes the trajectory of the particle , $x^\mu = x^\mu(\tau)$. Varying the path as

$$x^\mu \to x^\mu + \delta x^\mu$$

has the following effect on the action

$$\begin{aligned}\delta S &= \int d\tau \frac{1}{2}\frac{1}{\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)^{\frac{1}{2}}}\left(\delta g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + 2g_{\mu\nu}\delta\left(\frac{dx^\mu}{d\tau}\right)\frac{dx^\nu}{d\tau}\right)\\ &= \int d\tau \frac{1}{ds/d\tau}\left(\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\delta x^\lambda \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{d}{d\tau}\left(\delta x^\mu\right)\frac{dx^\nu}{d\tau}\right)\\ &= \int d\tau \frac{d\tau}{ds}\left(\frac{ds}{d\tau}\right)^2\left(\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\delta x^\lambda \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + g_{\mu\nu}\frac{d}{ds}\left(\delta x^\mu\right)\frac{dx^\nu}{ds}\right)\\ &= \int ds\left(\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\delta x^\lambda \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + g_{\mu\nu}\frac{d}{ds}\left(\delta x^\mu\right)\frac{dx^\nu}{ds}\right) \end{aligned} \tag{2.40}$$

Integrating the second term by parts

$$\int ds \underbrace{\frac{d}{ds}\left(\delta x^\mu\right)}_{f'}\underbrace{g_{\mu\nu}\frac{dx^\nu}{ds}}_{g} = -\int ds\, \delta x^\mu\Big(\underbrace{\frac{dg_{\mu\nu}}{ds}}_{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\rho}{ds}}\frac{dx^\nu}{ds} + g_{\mu\nu}\frac{d^2x^\nu}{ds^2}\Big)$$

brings us to

$$\delta S = \int ds\delta x^\lambda\left(\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - \frac{\partial g_{\lambda\nu}}{\partial x^\rho}\frac{dx^\rho}{ds}\frac{dx^\nu}{ds} - g_{\lambda\nu}\frac{d^2x^\nu}{ds^2}\right) = 0.$$

Therefore

$$\begin{aligned}\left(\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - \frac{\partial g_{\lambda\nu}}{\partial x^\rho}\frac{dx^\rho}{ds}\frac{dx^\nu}{ds}\right) - g_{\lambda\nu}\frac{d^2x^\nu}{ds^2} &= 0\\ \frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\left(\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \frac{\partial g_{\lambda\nu}}{\partial x^\mu}\right) - g_{\lambda\nu}\frac{d^2x^\nu}{ds^2} &= 0.\end{aligned} \tag{2.41}$$

Using

$$0 = g_{\mu\nu;\lambda} = \frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \Gamma^\rho_{\mu\lambda}g_{\rho\nu} - \Gamma^\rho_{\lambda\nu}g_{\mu\rho} \tag{2.42}$$

we can rewrite the first term in (2.41) as

$$\begin{aligned}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\left(\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \frac{\partial g_{\lambda\nu}}{\partial x^\mu}\right) &= \frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\left[\frac{1}{2}\left(\Gamma^\rho_{\mu\lambda}g_{\rho\nu} + \Gamma^\rho_{\lambda\nu}g_{\mu\rho}\right) - \Gamma^\rho_{\lambda\mu}g_{\rho\nu} - \Gamma^\rho_{\mu\nu}g_{\lambda\rho}\right]\\ &= -\frac{1}{2}\Gamma^\rho_{\mu\lambda}g_{\rho\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + \frac{1}{2}\Gamma^\rho_{\lambda\nu}g_{\mu\rho}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - \Gamma^\rho_{\mu\nu}g_{\lambda\rho}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\\ &= -\Gamma^\rho_{\mu\nu}g_{\lambda\rho}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\end{aligned}$$

where the first two terms in the second line cancel. As a result, (2.41) becomes

$$\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\Gamma^\rho_{\mu\nu}g_{\lambda\rho} + g_{\lambda\nu}\frac{d^2x^\nu}{ds^2} = 0. \tag{2.43}$$

Relabeling the dummy indices ν with ρ, we arrive at

$$\boxed{\frac{d^2x^\nu}{ds^2} + \Gamma^\nu_{\mu\rho}\frac{dx^\mu}{ds}\frac{dx^\rho}{ds} = 0} \tag{2.44}$$

This so called *geodesic equation* describes the motion of a free-falling particle in a gravitational field which minimizes the action (if the particle is massive, this translates to maximal proper time).

Remark 2.6. The geodesic equation we just derived can be obtained from a simpler action

$$S = -\int d\tau g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{2.45}$$

where τ must now be the proper time. The above action is essentially the same as (2.39) apart from the square root. The fact the two actions yield the same equation of motion can be understood by observing that for any non-zero function f, we have $\delta\left(\sqrt{f}\right) \propto \delta f/\sqrt{f}$. Hence, by extremizing the above action, whose Lagrangian is the square of the Lagrangian of (2.39), we automatically extremize the original single-particle action (2.39).

Remark 2.7. We know that massless particles follow null paths, i.e. those with $ds = 0$. Consequently, the action (2.39) we considered earlier ceases to be useful. To account for the possibility that the particle in question is massless, we consider the following action

$$S = -\frac{1}{2}\int d\sigma \left[\eta(\sigma) g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} + \frac{m^2}{\eta(\sigma)}\right] \tag{2.46}$$

where σ is the time coordinate in a locally Minkowskian coordinates system and we have introduced a new parameter $\eta(\sigma)$, which will eventually disappear. One can check that demanding the above action to be stationary under a variation in $\eta(\sigma)$ yields

$$\eta(\sigma) = \frac{m}{\sqrt{g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}}} \tag{2.47}$$

which we can plug back in to (2.46) to give

$$S = -m\int \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}}\, d\sigma \tag{2.48}$$

i.e. the action (2.39) we wrote down earlier if we choose $\tau = \sigma$. In other words, the new action works well when the particle is massive. Now, if the particle is massless, we can set $m = 0$ in (2.46), leaving us with

$$S = -\frac{1}{2}\int d\sigma \left[\eta(\sigma) g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}\right]. \tag{2.49}$$

Varying it with respect to η gives the null condition $ds = 0$ and varying it with respect to x^μ gives the geodesic equation

$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} = 0. \tag{2.50}$$

2.6 GEODESIC EQUATION IN THE NEWTONIAN APPROXIMATION

One requirement for a successful theory of gravity is that it must reproduce the results of Newtonian gravity and mechanics in the weak-field, non-relativistic limit. Consider a non-relativistic ($ds \approx dt$) particle in a static gravitational field. The particle's geodesic equations (2.44) for $\mu = i$ reduce to

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{00} = 0 \tag{2.51}$$

after neglecting terms quadratic in dx^i/ds and substituting $dx^i/ds \to dx^i/dt$. Mathematically, the weak gravitational field condition means that the metric $g_{\mu\nu}$ can be written as a slightly perturbed Minkowski metric $\eta_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}, \quad \delta g_{\mu\nu} \ll 1. \tag{2.52}$$

The Christoffel symbols Γ^i_{00} can be computed explicitly using (1.106) to first order in $\delta g_{\mu\nu}$

$$\Gamma^i_{00} = \frac{1}{2}\eta^{i\chi}(\overbrace{\delta g_{0\chi,0} + \delta g_{\chi 0,0}}^{=0 \quad \text{static field}} - \delta g_{00,\chi}) = \frac{1}{2}\frac{\partial \delta g_{00}}{\partial x^i} \tag{2.53}$$

and so the geodesic equation becomes

$$\frac{d^2x^i}{dt^2} + \frac{1}{2}\frac{\partial(\delta g_{00})}{\partial x^i} = 0. \tag{2.54}$$

In this case, the equation of motion predicted by Newtonian gravity is

$$\frac{d^2\mathbf{x}}{dt^2} + \nabla\varphi = 0 \tag{2.55}$$

where the gravitational potential φ can, for instance, be taken as that due to a point particle $\varphi = -G\frac{M}{r}$. Comparing this with the geodesic equation (2.54), we find the following weak-field correspondence

$$g_{00} \approx 1 + 2\varphi. \tag{2.56}$$

2.7 GRAVITATIONAL REDSHIFT

Consider a clock in a static gravitational field and pick a coordinate system where the clock is at rest, i.e. where the clock has $dx^i = 0$. The fact that the gravitational field is static means the metric $g_{\mu\nu}$ does not depend on the time coordinate x^0. Since the clock is at rest, its tick rate $d\tau$ is given by

$$d\tau^2 = ds^2\Big|_{dx^i=0} = g_{00}(dx^0)^2 \tag{2.57}$$

and its reading (the proper time) is

$$\tau = \int \sqrt{g_{00}}\,dx^0 \tag{2.58}$$

i.e. the amount of time elapsed as measured by a clock depends on the local gravitational potential around the clock. However, we cannot measure the said effect locally (by performing measurements where the clock is located) because any time-measuring device placed near the clock of interest would slow down or speed up in exactly the same way.

Nevertheless, it is possible to observe the effect of gravitational field on the time flow rate by comparing the rates of two identical processes occurring at different positions with different gravitational potentials. For concreteness, let us consider identical substances undergoing identical atomic transitions at rest at two different points: point 1 and point 2. The substances are set up such that every time there is a transition a pulse of electromagnetic wave is emitted from the point where it occurs. Suppose that there is an observer located at point 1 trying to measure the time interval between two consecutive pulses coming from a common source. In particular, the observer wants to compare the interval between pulses coming from point 1 with that of the pulses coming from point 2[1].

In a static spacetime, we can have a well-defined global time coordinate x^0. It is the time coordinate of the coordinate system where the spacetime / gravitational field looks static. The spacetime interval between two pulses is frame independent. Suppose that this interval is known in the rest frame of the substance to be $d\tau$. The frame-independence of spacetime interval then tells us that

$$d\tau^2 = g_{00}(x_1)\left(dx_1^0\right)^2 = g_{00}(x_2)\left(dx_2^0\right)^2 \tag{2.59}$$

[1] If the observer measures at point 1 the rate of the pulses coming from point 1 and then moves to point 2 to measure the rate of pulses coming from point 2, then there would be no difference between the two measured rates.

where x_1^0 and x_2^0 are the global time interval at x_1 and x_2 respectively. Now, let us come back to the observer at point 1 we introduced earlier. The time interval dt_1 between pulses from point 1 measured by the observer is

$$dt_1 = d\tau. \tag{2.60}$$

On the other hand, by (2.59), consecutive pulses coming from point 2 are separated from one another by

$$dx_2^0 = \frac{d\tau}{\sqrt{g_{00}(x_2)}} \tag{2.61}$$

in the global time. At point 1, the time interval dt_2 between these pulses is measured by the observer as

$$dt_2 = \sqrt{g_{00}(x_1)}dx_2^0 = \sqrt{\frac{g_{00}(x_1)}{g_{00}(x_2)}}d\tau. \tag{2.62}$$

Therefore, the ratio of the frequencies of pulses coming from point 1 and point 2 as measured by the observer is

$$\frac{\nu_1}{\nu_2} = \frac{dt_2}{dt_1} = \sqrt{\frac{g_{00}(x_1)}{g_{00}(x_2)}}. \tag{2.63}$$

Note that this is an exact relation. We did not make any approximation in the derivation. In the weak field limit, we can use the correspondence (2.56) to write the above as

$$\frac{\nu_2}{\nu_1} = \sqrt{\frac{g_{00}(x_2)}{g_{00}(x_1)}} = \sqrt{\frac{1+2\varphi(x_2)}{1+2\varphi(x_1)}} \approx 1 + \varphi(x_2) - \varphi(x_1)$$

or

$$\frac{\nu_2 - \nu_1}{\nu_1} \approx \varphi(x_2) - \varphi(x_1) \tag{2.64}$$

i.e. the pulses coming from point 2 are redshifted relative to those coming from point 1 if the gravitational potential at point 2 is lower than that at point 1.

Example 2.1. Suppose that point 1 is on the surface of the Earth and point 2 is on the surface of the Sun. Since the gravitational potential on the surface of the Sun is lower (more negative) than that on the surface of the Earth, we expect the light emitted by the Sun to be redshifted when it is measured on Earth by the amount

$$\frac{\nu_E - \nu_\odot}{\nu_\odot} \approx \frac{G}{c^2}\left(\underbrace{\frac{M_\odot}{R_\odot + d_{SE}}}_{\approx 0} - \frac{M_\odot}{R_\odot}\right) \approx -\frac{GM_\odot}{c^2 R_\odot} \approx -2\times 10^{-6}.$$

Due to its smallness, this gravitational redshift effect is difficult to disentangle from Doppler effects.

2.8 PRINCIPLE OF GENERAL COVARIANCE

It is natural to think that the laws of physics are independent of the way we view them. We have seen by now different manifestations of this idea, e.g. in electrodynamics and special relativity. One basic premise of special relativity is the principle of Lorentz covariance, which states that the laws of physics are the same in all inertial coordinate systems. In general relativity, this principle is extended to include all coordinates systems and goes under the name of *general covariance*. Specifically, the principle of general covariance says that the laws of physics preserve their forms under a general coordinate transformation $x^\mu \to x'^\mu$. In practice, we effortlessly make sure that general covariance holds by limiting ourselves to tensorial equations. General covariance allows us to rewrite any Lorentz-invariant equation in an arbitrary coordinate system with an arbitrary metric, which is not necessarily related to a Minkowskian one by a coordinate transformation.

2.9 CURVATURE

2.9.1 Dynamical equation of spacetime and Riemann tensor

Thus far, we have devoted most of our attention to studying the motion of a particle in a curved spacetime. Our aim now is to figure out how the spacetime itself behaves, i.e. we are looking for an equation dictating the dynamics of the spacetime. Based on the clues we have gathered so far, such an equation must have the following properties

- The LHS and RHS of the equation must transform in the same way under coordinate transformations, i.e. it must be general covariant.
- Gravitational effects are encoded in the metric $g_{\mu\nu}$.
- In the weak-field limit, the equation must reproduce the Poisson equation $\nabla^2\varphi = G_N\rho$ with the following correspondence $g_{00} = 1 + 2\varphi$.

Based on these properties, we can guess that the equation of interest has the form

$$[\text{second order derivatives}]\, g_{\mu\nu} = T_{\mu\nu}. \tag{2.65}$$

If we take the above form for granted, what is left to be done is finding the explicit form of the left hand side. It must be a tensor involving second order derivatives of the metric $g_{\mu\nu}$. The challenge here is that, as we have showed before, $g_{\mu\nu;\lambda} = 0$.

The so-called *Riemann tensor* fits our criteria. At the very least, it is a tensor that involves second order derivatives of the metric $g_{\mu\nu}$ and is non-zero when the spacetime is curved. To construct the Riemann tensor, take a covariant vector field V_μ and compute the commutator of its covariant derivatives

$$V_{\mu;\nu;\rho} - V_{\mu;\rho;\nu}. \tag{2.66}$$

This quantity is identically zero if the spacetime is flat, but not necessarily so if the spacetime is curved. To compute the double covariant derivatives, starting from the first term, let us group $V_{\mu;\nu}$ together and call it a tensor $W_{\mu\nu}$. We know well how to compute the covariant derivative of such a tensor

$$\begin{aligned}
W_{\mu\nu;\rho} &= \frac{\partial W_{\mu\nu}}{\partial x^\rho} - \Gamma^\chi_{\rho\mu} W_{\chi\nu} - \Gamma^\chi_{\nu\rho} W_{\mu\chi} \\
&= \frac{\partial}{\partial x^\rho}\left(\frac{\partial V_\mu}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu} V_\lambda\right) - \Gamma^\chi_{\rho\mu}\left(\frac{\partial V_\chi}{\partial x^\nu} - \Gamma^\lambda_{\chi\nu} V_\lambda\right) - \Gamma^\chi_{\nu\rho}\left(\frac{\partial V_\mu}{\partial x^\chi} - \Gamma^\lambda_{\mu\chi} V_\lambda\right) \\
&= \underbrace{\frac{\partial^2 V_\mu}{\partial x^\rho \partial x^\nu}}_{\text{symmetric } \nu\leftrightarrow\rho} - \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\rho} V_\lambda - \underbrace{\Gamma^\lambda_{\mu\nu}\frac{\partial V_\lambda}{\partial x^\rho} - \Gamma^\chi_{\rho\mu}\frac{\partial V_\chi}{\partial x^\nu}}_{\text{symmetric } \nu\leftrightarrow\rho} - \underbrace{\Gamma^\chi_{\nu\rho}\frac{\partial V_\mu}{\partial x^\chi}}_{\text{symmetric } \nu\leftrightarrow\rho} \\
&\quad + (\Gamma^\chi_{\rho\mu}\Gamma^\lambda_{\chi\nu} + \underbrace{\Gamma^\chi_{\nu\rho}\Gamma^\lambda_{\mu\chi}}_{\text{sym } \nu\leftrightarrow\rho})V_\lambda.
\end{aligned} \tag{2.67}$$

The quantity we are interested in is the difference between two such covariant derivatives with the indices $\nu\rho$ antisymmetrized. All the terms that are symmetric in $\nu\rho$ will vanish, leaving us with

$$\begin{aligned} V_{\mu;\nu;\rho} - V_{\mu;\rho;\nu} &= \Gamma^{\chi}_{\rho\mu}\Gamma^{\lambda}_{\chi\nu}V_\lambda - \frac{\partial\Gamma^{\lambda}_{\mu\nu}}{\partial x^\rho}V_\lambda - \Gamma^{\chi}_{\nu\mu}\Gamma^{\lambda}_{\chi\rho}V_\lambda + \frac{\partial\Gamma^{\lambda}_{\mu\rho}}{\partial x^\nu}V_\lambda \\ &= \left(\frac{\partial\Gamma^{\lambda}_{\mu\rho}}{\partial x^\nu} - \frac{\partial\Gamma^{\lambda}_{\mu\nu}}{\partial x^\rho} + \Gamma^{\chi}_{\mu\rho}\Gamma^{\lambda}_{\chi\nu} - \Gamma^{\chi}_{\mu\nu}\Gamma^{\lambda}_{\chi\rho}\right)V_\lambda \\ &\equiv R^{\lambda}{}_{\mu\nu\rho}V_\lambda. \end{aligned} \tag{2.68}$$

In the last equality we have defined the Riemann tensor $R^{\lambda}{}_{\mu\rho\nu}$ we were after (we know that it is a tensor because the left hand side is a tensor). When its first index is lowered, this tensor is given by

$$\boxed{R_{\chi\mu\nu\rho} = g_{\chi\lambda}R^{\lambda}{}_{\mu\nu\rho} = g_{\chi\lambda}\left(\frac{\partial\Gamma^{\lambda}_{\mu\rho}}{\partial x^\nu} - \frac{\partial\Gamma^{\lambda}_{\mu\nu}}{\partial x^\rho} + \Gamma^{\chi}_{\mu\rho}\Gamma^{\lambda}_{\chi\nu} - \Gamma^{\chi}_{\mu\nu}\Gamma^{\lambda}_{\chi\rho}\right).} \tag{2.69}$$

Using (1.106) we can rewrite it as

$$\boxed{R_{\lambda\mu\chi\nu} = \frac{1}{2}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^\chi \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\chi \partial x^\lambda} - \frac{\partial^2 g_{\lambda\chi}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\chi}}{\partial x^\nu \partial x^\lambda}\right) + g_{\eta\sigma}\left(\Gamma^{\eta}_{\nu\lambda}\Gamma^{\sigma}_{\mu\chi} - \Gamma^{\eta}_{\chi\lambda}\Gamma^{\sigma}_{\mu\nu}\right)} \tag{2.70}$$

clearly showing that it involves second-order derivatives of $g_{\mu\nu}$, as claimed earlier. From the Riemann tensor, we can construct two more useful quantities: the *Ricci tensor* and the *Ricci scalar*. The Ricci tensor $R_{\mu\kappa}$, the only (0,2) tensor (up to a constant factor) we can construct from the Riemann tensor, is obtained by contracting the first and third index of the Riemann tensor

$$\boxed{R_{\mu\kappa} \equiv R^{\lambda}{}_{\mu\lambda\kappa}} \tag{2.71}$$

and the *Ricci scalar* R, the only scalar (up to a constant factor) we can construct from the Riemann tensor, is obtained by contracting the sole two indices of the Ricci tensor

$$\boxed{R \equiv g^{\mu\nu}R_{\mu\nu}.} \tag{2.72}$$

Remark 2.8. Symmetry properties of the Riemann tensor $R_{\lambda\mu\nu\chi}$:

- Symmetry under simultaneous exchanges of the $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$ indices :

$$R_{\lambda\mu\nu\chi} = R_{\nu\chi\lambda\mu}. \tag{2.73}$$

- Antisymmetry under an exchange of $1 \leftrightarrow 2$ or $3 \leftrightarrow 4$ indices:

$$R_{\lambda\mu\nu\chi} = -R_{\mu\lambda\nu\chi} = R_{\mu\lambda\chi\nu} = -R_{\lambda\mu\chi\nu}. \tag{2.74}$$

- Cyclic property on the last three indices

$$R_{\lambda\mu\nu\chi} + R_{\lambda\chi\mu\nu} + R_{\lambda\nu\chi\mu} = 0. \tag{2.75}$$

Remark 2.9. Number of independent components of $R_{\mu\nu\rho\sigma}$.
The indices $\mu\nu$ are antisymmetric, so they represent $(16-4)/2 = 6$ independent components; the same goes for $\rho\sigma$. For the present counting purposes, we can take $\mu\nu$ as a single index with 6 possible values. Again, the same applies to $\rho\sigma$. Now, $\mu\nu$ and $\rho\sigma$ are symmetric. So, at this point, we count $(6.6-6)/2+6 = 21$ independent components. Next, the cyclic property of $R_{\mu\nu\rho\sigma}$ in its last three indices is worth 1 degree of freedom of constraint. Hence, we are now left with $21-1 = 20$ independent components, that is, **algebraically** independent components. At a particular

point, we still have a freedom to perform local Lorentz transformation, which is parameterized by 6 parameters. So, in the end, there are only $20-6=14$ **physically** independent components that are invariant under Lorentz transformations, meaning that there are 14 Lorentz-invariant scalars that we can form from $R_{\mu\nu\rho\sigma}$.

If $R_{\mu\nu}=0$, i.e. a condition that is worth $(4.4-4)/2+4=10$ degrees of freedom, there are only $20-10=10$ algebraically independent components and $10-6=4$ physically independent components of the Riemann tensor.

2.9.2 Necessary and sufficient conditions for a flat spacetime

How do we identify if a spacetime is flat? Due to general covariance, the flatness of spacetime has a wider meaning than having a Minkowskian metric $\eta_{\mu\nu}$. Any spacetime whose metric $g_{\mu\nu}$ is related to $\eta_{\mu\nu}$ via a coordinate transform $\xi^\alpha(x)$

$$\eta^{\alpha\beta}=\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}g^{\mu\nu} \tag{2.76}$$

is also a flat spacetime. The necessary and sufficient conditions for a flat spacetime are:

1. $R^\lambda{}_{\mu\nu\chi}=0$ everywhere.
2. For all x, the metric tensor $g_{\mu\nu}(x)$ has one positive and three negative eigenvalues.

The first condition is obvious because we know that $R^\lambda{}_{\mu\nu\chi}=0$ in the coordinate system where the metric is equal to $\eta_{\mu\nu}$ and, since $R^\lambda{}_{\mu\nu\chi}$ is a tensor, it must be zero in any other coordinates. The second condition follows from a linear algebra theorem which says if the matrix G is related to the Minkowski matrix η by a transformation $G=D^T\eta D$, with $\det D\neq 0$, then G and η have the same number of positive, negative and zero eigenvalues.

2.9.3 Parallel transport and Riemann tensor

In a flat spacetime, the notion of parallel transport has a clear meaning: a vector V^μ is parallel transported along a curve $x^\mu(\tau)$ if the vector remains unchanged along the curve (see Figure 1)

$$\frac{dV_\alpha}{d\tau}=0 \quad\Longrightarrow\quad V_{\alpha,\mu}\frac{dx^\mu}{d\tau}=0. \tag{2.77}$$

This condition can be generalized to curved spacetimes as

$$V_{\alpha;\mu}\frac{dx^\mu}{d\tau}=0 \quad\Longrightarrow\quad \left(V_{\alpha,\mu}-\Gamma^\chi_{\alpha\mu}V_\chi\right)\frac{dx^\mu}{d\tau}=0 \tag{2.78}$$

or

$$\boxed{\frac{dV_\alpha}{d\tau}=\Gamma^\chi_{\alpha\mu}V_\chi\frac{dx^\mu}{d\tau}.} \tag{2.79}$$

Physically, one can carry out parallel transport in curved spacetime in the following way. At each point along the path, we can go to the local flat coordinate system. In this coordinate system, parallel transport has a clear definition and we can proceed infinitesimally according to this definition.

One way to check if a spacetime is flat or curved is by parallel transporting a vector around a closed loop. If the spacetime is flat then we would get at the end the same vector we started with. If the spacetime is curved in general we will get a different vector. To understand this better, let us work out explicitly the result of

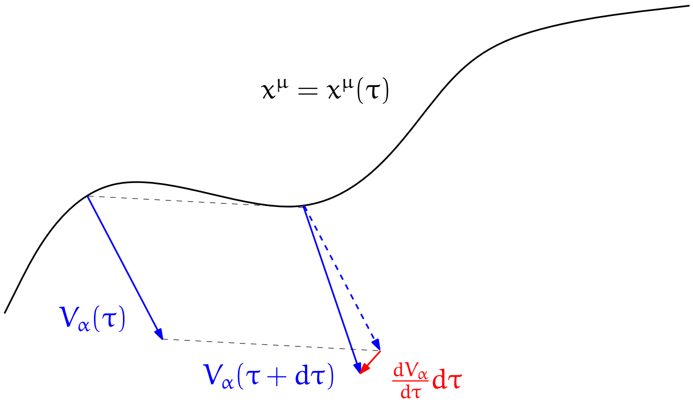

Figure 1: Parallel transport on a Euclidean space

parallel transporting a vector around a closed loop. The total change ΔV_α in the transported vector V_α can be obtained by integrating (2.79)

$$\Delta V_\alpha = \oint \Gamma^\chi_{\alpha\mu} V_\chi \frac{dx^\mu}{d\tau} d\tau. \tag{2.80}$$

Consider a small loop connecting the points x_0^μ, $x_0^\mu + da^\mu$, $x_0^\mu + da^\mu + db^\mu$, and $x_0^\mu + db^\mu$. Along this loop, the vector V_λ and the Christoffel symbols can be Taylor expanded as

$$\begin{aligned} V_\chi(x) &= V_\chi(0) + \frac{dV_\chi}{dx^\mu}(0)(x^\mu - x_0^\mu) + O(x^\mu - x_0^\mu)^2 \\ &= V_\chi(0) + \Gamma^\lambda_{\chi\mu} V_\lambda(0)(x^\mu - x_0^\mu) + O(x^\mu - x_0^\mu)^2 \end{aligned} \tag{2.81}$$

$$\Gamma^\chi_{\alpha\mu}(x) = \Gamma^\chi_{\alpha\mu}(0) + \frac{\partial \Gamma^\chi_{\alpha\mu}}{\partial x^\nu}(0)(x^\nu - x_0^\nu) + O(x^\nu - x_0^\nu)^2. \tag{2.82}$$

Injecting these to (2.80), we get

$$\begin{aligned} \Delta V_\alpha &= \underbrace{\oint \Gamma^\chi_{\alpha\mu}(0) V_\chi(0) \frac{dx^\mu}{\cancel{d\tau}} \cancel{d\tau}}_{=0 \text{ since } \oint dx^\mu = 0} + \oint \left(\frac{\partial \Gamma^\chi_{\alpha\mu}}{\partial x^\nu}(0) + \Gamma^\chi_{\lambda\nu}(0)\Gamma^\lambda_{\alpha\mu}(0) \right)(x^\nu - x_0^\nu) V_\chi(0) \frac{dx^\mu}{\cancel{d\tau}} \cancel{d\tau} \\ &= \left(\frac{\partial \Gamma^\chi_{\alpha\mu}}{\partial x^\nu}(0) + \Gamma^\chi_{\lambda\nu}(0)\Gamma^\lambda_{\alpha\mu}(0) \right) V_\chi(0) \underbrace{\oint (x^\nu - x_0^\nu) dx^\mu}_{= \oint x^\nu dx^\mu - x_0^\nu \oint dx^\mu} \\ &= \left(\frac{\partial \Gamma^\chi_{\alpha\mu}}{\partial x^\nu}(0) + \Gamma^\chi_{\lambda\nu}(0)\Gamma^\lambda_{\alpha\mu}(0) \right) V_\chi(0) \oint x^\nu dx^\mu. \end{aligned} \tag{2.83}$$

Integration by parts gives

$$\begin{aligned} \oint x^\nu dx^\mu &= \oint d\tau x^\nu \frac{dx^\mu}{d\tau} = \underbrace{\left| x^\mu(\tau) x^\nu(\tau) \right|_0^0}_{=0} - \oint d\tau x^\mu \frac{dx^\nu}{d\tau} \\ &= -\oint x^\mu dx^\nu \end{aligned}$$

i.e. $\oint x^\nu dx^\mu$ is antisymmetric in $\mu\nu$. Using this fact and the definition of Riemann tensor, we can rewrite (2.83) as

$$\Delta V_\alpha = \frac{1}{2} R^\chi{}_{\alpha\nu\mu}(0) V_\chi(0) \underbrace{\oint x^\nu dx^\mu}_{\neq 0}. \tag{2.84}$$

In conclusion, parallel transporting a vector around a closed infinitesimal loop results in no change in the vector unless $R^\chi{}_{\alpha\mu\nu} \neq 0$ at that point. This finding can be generalized to the case of an arbitrary finite closed loop by noting that a finite-loop integral can always be written as a sum of many infinitesimal-loop integrals. In order to cover the full area of the finite loop, the infinitesimal loops must overlap one another, and the contributions from those overlapping parts cancel as they are line integrals in opposite directions, but otherwise the same. The remaining part of the line integrals that do not get canceled are those that lie on the original finite closed loop, as intended. Therefore, the same qualitative conclusion (that parallel transporting a vector around a closed loop in a curved spacetime in general yields a different vector from what we started with) applies regardless of the size of the loop.

2.9.4 Bianchi identity

The Riemann tensor $R_{\lambda\mu\nu\kappa}$ obeys yet another important identity called the *Bianchi identity*, which reads

$$\boxed{R_{\lambda\mu\nu\chi;\eta} + R_{\lambda\mu\eta\nu;\chi} + R_{\lambda\mu\chi\eta;\nu} = 0} \tag{2.85}$$

where we have summed over the cyclic permutation of the last three indices. Before proving this identity, let us derive a useful consequence of the Bianchi identity. First, contract the identity with $g^{\nu\lambda}$ to get

$$\begin{aligned} 0 &= g^{\nu\lambda}\left(R_{\lambda\mu\nu\chi;\eta} + R_{\lambda\mu\eta\nu;\chi} + R_{\lambda\mu\chi\eta;\nu}\right) \\ &= R^{\lambda}{}_{\mu\lambda\chi;\eta} + \underbrace{R^{\lambda}{}_{\mu\eta\lambda;\chi}}_{=-R^{\lambda}{}_{\mu\lambda\eta;\chi}} + R^{\lambda}{}_{\mu\chi\eta;\lambda} \\ &= R^{\lambda}{}_{\mu\lambda\chi;\eta} - R^{\lambda}{}_{\mu\lambda\eta;\chi} + R^{\lambda}{}_{\mu\chi\eta;\lambda} \end{aligned} \tag{2.86}$$

which can be expressed in terms of Ricci tensors

$$R_{\mu\chi;\eta} - R_{\mu\eta;\chi} + R^{\lambda}{}_{\mu\chi\eta;\lambda} = 0. \tag{2.87}$$

Then, we contract the above once more with $g^{\mu\chi}$ to obtain

$$\begin{aligned} 0 &= g^{\mu\chi}\left(R_{\mu\chi;\eta} - R_{\mu\eta;\chi} + R^{\lambda}{}_{\mu\chi\eta;\lambda}\right) \\ &= R_{;\eta} \underbrace{- R^{\mu}{}_{\eta;\mu} - R^{\lambda}{}_{\eta;\lambda}}_{=-2R^{\mu}{}_{\eta;\mu}} \\ &= R_{;\eta} - 2R^{\mu}{}_{\eta;\mu} \end{aligned} \tag{2.88}$$

or

$$\left(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R\right)_{;\mu} = 0. \tag{2.89}$$

Finally, we obtain the following expression by multiplying with $g^{\nu\eta}$

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{;\mu} = 0. \tag{2.90}$$

The expression inside the bracket

$$\boxed{G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R} \tag{2.91}$$

is known as the *Einstein tensor*. We will find it useful when writing the gravitational field equation.

The strategy we are going to follow in proving the Bianchi identity (2.85) is to show that it is satisfied in a local Minkowski coordinate system and then argue that it must be valid in any coordinates because of its tensor form. Thanks to the equivalence principle, for an arbitrary point x_0, we can in principle go to the local Minkowskian coordinate system where the following equations hold

$$\begin{aligned} g_{\mu\nu}(x_0) &= \eta_{\mu\nu} \\ \Gamma^{\mu}_{\nu\rho}(x_0) &= 0. \end{aligned} \tag{2.92}$$

With the Christoffel symbols being zero, the covariant derivatives reduce to mere partial derivatives and hence the covariant derivative of the Riemann tensor reduces to

$$R_{\lambda\mu\chi\nu;\eta} = R_{\lambda\mu\nu\chi,\eta} = \frac{1}{2}\left(\frac{\partial^3 g_{\lambda\nu}}{\partial x^{\chi}\partial x^{\mu}\partial x^{\eta}} - \frac{\partial^3 g_{\mu\nu}}{\partial x^{\lambda}\partial x^{\chi}\partial x^{\eta}} - \frac{\partial^3 g_{\lambda\chi}}{\partial x^{\nu}\partial x^{\mu}\partial x^{\eta}} + \frac{\partial^3 g_{\mu\chi}}{\partial x^{\nu}\partial x^{\lambda}\partial x^{\eta}}\right)$$

where we have used (2.70). If we sum the above expression over the cyclic permutation in the last three indices, $\mu\nu\chi$, we will get zero, thus proving the Bianchi identity. Since this is a tensor equation and we started at an arbitrary point x_0, the result is valid in any coordinates at any point.

Remark 2.10. Local Minkowskian coordinates.
We can go from an arbitrary coordinate system to the local Minkowskian coordinates by the following procedure:

- Start in an arbitrary coordinate system x^{μ} with metric $g_{\mu\nu}$ and calculate the corresponding Christoffel symbols $\Gamma^{\mu}_{\nu\rho}(x_0)$.

- Introduce new coordinates x'^{μ} that are related to the ones we started with as

$$x'^{\mu} = x^{\mu} + \underbrace{\frac{1}{2}\Gamma^{\mu}_{\nu\rho}(x-x_0)^{\nu}(x-x_0)^{\rho}}_{=0 \text{ for } x=x_0}. \tag{2.93}$$

  Since $\partial x'^{\mu}/\partial x^{\nu} = \delta^{\mu}_{\nu}$ at x_0, this transformation keeps the $g_{\mu\nu}$ at x_0 unchanged. One can check that the Christoffel symbols in the new coordinate system are zero, $\Gamma'^{\mu}_{\nu\rho} = 0$.

- Diagonalize the metric so that it has the form

$$g''_{\mu\nu} = \text{diag}\,(-\lambda_0, \lambda_1, \lambda_2, \lambda_3) \tag{2.94}$$

  by making the coordinate transformation

$$x''^{\mu} = O^{\mu}_{\nu} x'^{\nu} \tag{2.95}$$

  where O is an orthogonal matrix.

- Finally, rescale the coordinates as

$$\tilde{x}^{\mu} = \frac{1}{\sqrt{\lambda^{\mu}}} x''^{\mu} \tag{2.96}$$

  to get what we are after

$$\begin{aligned} \tilde{g}_{\mu\nu}(x_0) &= \eta_{\mu\nu} \\ \tilde{\Gamma}^{\mu}_{\nu\rho}(x_0) &= 0. \end{aligned} \tag{2.97}$$

Note however that in a general spacetime, there is no way to set simultaneously the Christoffel symbols and their derivatives to zero, even at a given point. This comes from the fact that the Riemann tensor is indeed a tensor and cannot be made to vanish in a given coordinate system (as this would imply that it vanishes in any coordinate system).

2.10 ACTION PRINCIPLE FOR FIELDS

In analytical mechanics, one sees how to formulate the principle of least action for a discrete set of generalized coordinates q_i and generalized velocities $\dot{q}_i$. In describing gravity, the physical objects of interest are fields instead of discrete coordinates. Therefore it is of interest to formulate the principle of least action for fields. A field effectively assigns a generalized coordinate $q_\mathbf{x}$ and a generalized velocity $\dot{q}_\mathbf{x}$ to each spacetime point $\mathbf{x}$. The generalized coordinates may come in different forms, e.g. scalar, vector, or tensor. We write scalar-, vector-, and tensor-valued fields as $\varphi(x^\mu)$, $A^\mu(x^\nu)$, and $\varphi_{\mu\nu}(x^\alpha)$ respectively.

2.10.1 Scalar field in a flat spacetime

For simplicity, let us start by formulating the least action principle for a scalar field $\varphi(\mathbf{x})$ in a flat spacetime. Given the boundary conditions

$$\begin{aligned}\varphi(\mathbf{x},\mathbf{t_1}) &= \varphi_1(\mathbf{x})\\ \varphi(\mathbf{x},\mathbf{t_2}) &= \varphi_2(\mathbf{x}),\end{aligned} \tag{2.98}$$

the field $\varphi(\mathbf{x})$ evolves in such a way that the action

$$S = \int d^4x \mathcal{L}(\varphi, \partial\varphi) \tag{2.99}$$

is minimized. If we perturb the field ϕ slightly around a solution $\varphi_0(\mathbf{x}, t)$ as

$$\varphi(\mathbf{x}, t) = \varphi_0(\mathbf{x}, t) + \delta\varphi(\mathbf{x}, t) \tag{2.100}$$

where $\delta\varphi$ must obey the boundary conditions

$$\delta\varphi(\mathbf{x}, t_1) = \delta\varphi(\mathbf{x}, t_2) = 0 \tag{2.101}$$

and the locality condition

$$\delta\varphi(\mathbf{x}, t) \to 0, \quad \text{as } \mathbf{x} \to \infty \tag{2.102}$$

then the action does not change if

$$\begin{aligned}\delta S &= \int d^4x \left[\mathcal{L}\left(\varphi_0 + \delta\varphi, \partial_\mu\varphi_0 + \partial_\mu\delta\varphi\right) - \mathcal{L}\left(\varphi_0, \partial_\mu\varphi\right)\right]\\ 0 &= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi\right)}\partial_\mu\delta\varphi\right]\\ 0 &= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi\right)}\right)\delta\varphi\right]\\ &\quad + \underbrace{\int d^3x \left(\frac{\partial\mathcal{L}}{\partial\left(\partial_0\varphi\right)}\delta\varphi\right)\Bigg|_{t_1}^{t_2}}_{=0 \text{ boundary cond.}} + \underbrace{\oint dt d\mathbf{S}. \left(\frac{\partial\mathcal{L}}{\partial\left(\partial\varphi\right)}\delta\varphi\right)}_{=0 \text{ locality}}\end{aligned}$$

which is the *Euler-Lagrange equation* for fields

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) = 0. \tag{2.103}$$

Example 2.2. Consider the following action

$$S = \int d^4x\Big(F(\varphi) + \underbrace{\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi}_{\text{scalar kinetic term}}\Big) \tag{2.104}$$

with the scalar potential

$$F(\varphi) = -\frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4}\varphi^4. \tag{2.105}$$

Varying the field $\varphi(\mathbf{x},t) = \varphi_0(\mathbf{x},t) + \delta\varphi(\mathbf{x},t)$ around the solution $\phi_0(\mathbf{x},t)$ results in the variation of the action

$$\begin{aligned}
\delta S &= S[\varphi_0 + \delta\varphi] - S[\varphi_0] \\
0 &= \int d^4x \left(\frac{1}{2}\partial_\mu(\varphi_0+\delta\varphi)\partial^\mu(\varphi_0+\delta\varphi) - \frac{1}{2}m^2(\varphi_0+\delta\varphi)^2 - \frac{\lambda}{4}(\varphi_0+\delta\varphi)^4\right) - S[\varphi_0] \\
0 &= \cancel{S[\varphi_0]} - \int d^4x \left(m^2\varphi_0\delta\varphi + \lambda\varphi_0^3\delta\varphi - \partial_\mu\varphi_0\partial^\mu\delta\varphi\right) - \cancel{S[\varphi_0]} + O(\delta\phi^2) \\
0 &= \int d^4x \left(\partial_\mu\varphi_0\partial^\mu\delta\varphi - m^2\varphi_0\delta\varphi - \lambda\varphi_0^3\delta\varphi\right) + O(\delta\phi^2).
\end{aligned} \tag{2.106}$$

Integrating the first term by parts and using the locality condition, we get

$$0 = \int d^4x \left(\partial^\mu\partial_\mu\varphi_0 + m^2\varphi_0 + \lambda\varphi_0^3\right)\delta\varphi \tag{2.107}$$

or

$$\Box\varphi(\mathbf{x},t) + m^2\varphi(\mathbf{x},t) + \lambda\varphi^3(\mathbf{x},t) = 0. \tag{2.108}$$

Alternatively, we could have arrived at the equation of motion above directly, by using the Euler-Lagrange equation.

Example 2.3. Lagrangian formulation of electrodynamics.
The free (without sources) Maxwell equations can be obtained from the following action

$$S = -\int d^4x \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.109}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.110}$$

Varying the action and setting it to zero

$$\begin{aligned}
\delta_A S =& S[A^\mu + \delta A^\mu] - S[A^\mu] \\
0 =& \int d^4x \Big[\cancel{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}} - \frac{1}{4}\left(F_{\mu\nu}\delta F^{\mu\nu} + F^{\mu\nu}\delta F_{\mu\nu}\right) + \cancel{\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}\Big] \\
0 =& -\int d^4x \frac{1}{2}F_{\mu\nu}\delta F^{\mu\nu} \\
0 =& -\int d^4x \frac{1}{2}F_{\mu\nu}\left(\partial^\mu\delta A^\nu - \partial^\nu\delta A^\mu\right).
\end{aligned} \tag{2.111}$$

Since $F_{\mu\nu}$ is antisymmetric, we can rewrite the above as

$$0 = \int d^4x \left(-F_{\mu\nu}\partial^\mu\delta A^\nu\right). \tag{2.112}$$

Integrating by parts and imposing the locality condition gives

$$0 = \int d^4x \left(\partial^\mu F_{\mu\nu}\right)\delta A^\nu$$

i.e. we arrive at the *homogeneous Maxwell equations* (Gauss law and Ampere's law)

$$\partial^\mu F_{\mu\nu} = 0. \tag{2.113}$$

The remaining two equations (magnetic Gauss law and Faraday's law) are obtained automatically from the symmetry property of the field strength tensor $F_{\mu\nu}$

$$\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 0. \tag{2.114}$$

2.10.2 Scalar field in a curved spacetime

The action for a scalar field in a flat spacetime can be generalized to its curved-spacetime version by making the following replacements:

- $d^4x \to \sqrt{-g}d^4x$ in the action.
- $\partial_\mu \to \nabla_\mu$ (or $, \to ;$) in the Lagrangian and equations of motion.
- $\eta_{\mu\nu} \to g_{\mu\nu}$ in the Lagrangian and equations of motion.

For example, the curved-spacetime version of the action with the scalar potential considered in the previous section is

$$S = \int \sqrt{-g}d^4x \left(\frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{,\nu} - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4\right) \tag{2.115}$$

and the corresponding equation of motion is

$$\phi^{;\mu}{}_{;\mu} - m^2\phi - \lambda\phi^3 = 0. \tag{2.116}$$

For another example, the curved-spacetime generalization of the free electrodynamics action is

$$S = \int \sqrt{-g}d^4x \left(-\frac{1}{4}g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}\right) \tag{2.117}$$

and its equation of motion becomes

$$F_{\mu\nu}{}^{;\nu} = 0. \tag{2.118}$$

2.10.3 Einstein-Hilbert action and Einstein equation

We are now ready to formulate the least action principle for gravity. The action describing gravity must be a scalar, as any action should be, and a function of the metric $g_{\mu\nu}$, as it is meant to describe gravity. The simplest of such actions is called the *Einstein-Hilbert action*. It reads

$$S_E = \int d^4x\sqrt{-g}\left(-\frac{1}{16\pi G}R - \frac{\lambda}{8\pi G}\right) \tag{2.119}$$

where R is the Ricci scalar we introduced earlier and λ is a constant dubbed *cosmological constant*. It is not difficult to construct more complicated scalars out of the metric. To the Einstein-Hilbert action above, we could add various contractions of the Riemann tensor or Ricci tensor, e.g. $R_{\mu\nu}R^{\mu\nu}$. However, as we will see, in most cases of interest such terms are suppressed, i.e. their contributions to the dynamics are negligible.

The *Einstein equations*, our current best description of gravity, follow from varying the Einstein-Hilbert action with respect to the metric $g_{\mu\nu}$. Before we proceed to derive the Einstein equations, we would like to derive a few useful identities related to the variation of the metric $g_{\mu\nu}$. The variation of the metric $\delta g_{\mu\nu}$ and the variation of its inverse $\delta g^{\mu\nu}$ are related via

$$g_{\mu\nu}g^{\nu\rho} = \delta_\mu{}^\rho \quad \Longrightarrow \quad g_{\mu\nu}\delta g^{\nu\rho} + \delta g_{\mu\nu}g^{\nu\rho} = 0 \tag{2.120}$$

Next, we are going to work out the expression for the variation of determinant of the metric $g \equiv \det g$. Recall the following identity from linear algebra

$$g = \exp\left(\mathrm{Tr}\log g_{\mu\nu}\right). \tag{2.121}$$

Varying this identity gives

$$\begin{aligned}
g + \delta g &= \exp \mathrm{Tr}\left[\ln(g_{\mu\nu} + \delta g_{\mu\nu})\right] \\
&= \exp \mathrm{Tr}\left[\ln(g_{\mu\nu}) + g^{-1}_{\mu\nu}\delta g_{\mu\nu}\right] \\
&= \underbrace{e^{\mathrm{Tr}(g_{\mu\nu})}}_{=g} e^{\mathrm{Tr}(g^{-1}_{\mu\nu}\delta g_{\mu\nu})} \qquad (2.122) \\
&= g\left(1 + \mathrm{Tr}\left(g^{\mu\nu}\delta g_{\mu\nu}\right)\right) \\
&= g\left(1 - g_{\mu\nu}\delta g^{\mu\nu}\right) \qquad (2.123)
\end{aligned}$$

where we have used the fact that $g^{\mu\nu}\delta g_{\mu\nu}$ is a scalar and used the result (2.120) in the last step. Therefore we can see that (as we did in the third problem in Problem Set 4)

$$\delta g = -g g_{\mu\nu}\delta g^{\mu\nu}. \qquad (2.124)$$

The identity (2.124) allows us to evaluate the variation of the $\sqrt{-g}$ appearing in the Einstein-Hilbert action

$$\delta\left(\sqrt{-g}\right) = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \qquad (2.125)$$

Armed with the above identities, we can now vary the Einstein-Hilbert action starting from the Ricci scalar R term

$$\begin{aligned}
\delta\int d^4x\sqrt{-g}R &= \delta\int d^4x\sqrt{-g}g^{\mu\nu}R_{\mu\nu} \\
&= \int d^4x\delta(\sqrt{-g})g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}\delta(g^{\mu\nu}R_{\mu\nu}) \\
&\overset{(2.125)}{=} \int d^4x\sqrt{-g}\left(-\frac{1}{2}g_{\alpha\beta}\delta g^{\alpha\beta}\underbrace{g^{\mu\nu}R_{\mu\nu}}_{=R} + \delta(g^{\mu\nu})R_{\mu\nu} + g^{\mu\nu}\delta(R_{\mu\nu})\right) \\
&= \int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} + \int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}.
\end{aligned}$$
$$(2.126)$$

To compute the second integral, we go to the local Minkowskian coordinates where $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^{\rho}_{\mu\nu} = 0$. In these coordinates, the Ricci tensor can be written in the form

$$R_{\mu\nu} = \frac{\partial\Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} - \frac{\partial\Gamma^{\alpha}_{\mu\alpha}}{\partial x^{\nu}}. \qquad (2.127)$$

and so the expression $g^{\mu\nu}\delta R_{\mu\nu}$ becomes

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\left(\frac{\partial\delta\Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} - \frac{\partial\delta\Gamma^{\alpha}_{\mu\alpha}}{\partial x^{\nu}}\right) \overset{g_{\mu\nu}=\eta_{\mu\nu}}{=} \frac{\partial}{\partial x^{\lambda}}\left(\underbrace{g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^{\alpha}_{\mu\alpha}}_{=\omega^{\lambda}}\right) = \frac{\partial\omega^{\lambda}}{\partial x^{\lambda}}$$

and hence

$$\begin{aligned}
\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} &= \int d^4x\sqrt{-g}\frac{\partial\omega^{\lambda}}{\partial x^{\lambda}} \\
&\overset{g_{\mu\nu}=\eta_{\mu\nu}}{=} \underbrace{\int d^4x\frac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}\omega^{\lambda}\right)}_{\text{divergence theorem}} \\
&= \oint_S dS_{\lambda}\sqrt{-g}\omega^{\lambda} = 0
\end{aligned}$$

since $\omega^{\lambda} = 0$ at infinity. Therefore, the remaining term is

$$\delta\int d^4x\sqrt{-g}R = \int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu}. \qquad (2.128)$$

Notice that the expression inside the brackets is the Einstein tensor $G_{\mu\nu}$ defined in (2.91). Furthermore, the variation of the cosmological constant term in (2.119) is

$$\delta \int d^4x\sqrt{-g}\frac{\lambda}{8\pi G} = \frac{\lambda}{8\pi G}\int d^4x\left(-\frac{1}{2\sqrt{-g}}\delta g\right) = -\frac{1}{2}\int d^4x\left(\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\right). \tag{2.129}$$

Combining (2.128) and (2.129), we find that the variation of the Einstein-Hilbert action is given by

$$\delta S_E = -\int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \lambda g_{\mu\nu}\right)\delta g^{\mu\nu}\frac{1}{16\pi G}. \tag{2.130}$$

Suppose that in addition to the Einstein-Hilbert action S_E, we also have an action S_M describing the behavior of matter. The variation of S_M with respect to $g_{\mu\nu}$ can be written as

$$\delta S_M = \int dx^4\sqrt{-g}\left[\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}\delta g^{\mu\nu}\right]. \tag{2.131}$$

Finally, demanding that the variation of the total action is zero

$$\delta S_E + \delta S_M = 0 \tag{2.132}$$

we obtain the Einstein equation

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}} \tag{2.133}$$

where we have defined the *energy-momentum tensor* $T_{\mu\nu}$ as

$$\boxed{T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}.} \tag{2.134}$$

The energy-momentum conservation

$$T_{\mu\nu;\mu} = 0 \tag{2.135}$$

follows from $g_{\mu\nu;\rho} = 0$ and the Bianchi identity in a way analogous to the current conservation in electrodynamics $(\partial_\mu F^{\mu\nu} = j^\nu) \to \partial_\nu j^\nu = 0$.

Remark 2.11. Canonical energy-momentum tensor and Hilbert energy-momentum tensor.

You might have seen in the Quantum Field Theory lectures a different definition of the energy-momentum tensor in flat spacetime

$$T_C^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L} \tag{2.136}$$

that derives from the Noether theorem in the presence of translational symmetry. We call this version of the energy-momentum tensor *canonical energy-momentum tensor*, in contrast to the one we defined earlier (2.134), which is known as the *Hilbert energy-momentum tensor*. While the Hilbert energy-momentum tensor is symmetric in $\mu\nu$ and gauge-invariant, the canonical energy-momentum (2.136) is in general not so. The two versions of energy-momentum tensor can in general be related as

$$T^{\mu\nu} = T_C^{\mu\nu} + \frac{1}{2}\partial_\lambda\psi^{\lambda\mu\nu} \tag{2.137}$$

where $\psi^{\lambda\mu\nu}$ is an appropriate antisymmetric tensor with the property

$$\psi^{\lambda\mu\nu} = -\psi^{\mu\lambda\nu}. \tag{2.138}$$

Thanks to this property, we have

$$\partial_\mu\partial_\lambda\psi^{\lambda\mu\nu} = 0 \tag{2.139}$$

and consequently adding $\partial_\lambda\psi^{\lambda\mu\nu}$ to the canonical stress-energy energy tensor does not change the fact that it is conserved.

Example 2.4. Energy momentum tensor for a scalar field.
A free scalar field in curved spacetime is described by the following action

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}m^2\varphi^2\right). \tag{2.140}$$

We would like to compute the energy-momentum tensor for this action using (2.134). Varying the action with respect to $g_{\mu\nu}$ yields

$$\begin{aligned}\delta S_g &= \int d^4x\delta(\sqrt{-g})\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}m^2\varphi^2\right) + \int d^4x\sqrt{-g}\left(\frac{1}{2}\delta g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right)\\ &= \int d^4x\frac{1}{2}\sqrt{-g}\left(\partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\mathcal{L}\right)\delta g_{\mu\nu}\end{aligned} \tag{2.141}$$

where we have used Eq. (2.125). Therefore the stress-energy tensor is

$$T_{\mu\nu} = \frac{\partial\varphi}{\partial x^\mu}\frac{\partial\varphi}{\partial x^\nu} - g_{\mu\nu}\mathcal{L} \tag{2.142}$$

which has the same form as in a flat space apart from the substitution $\eta_{\mu\nu} \to g_{\mu\nu}$.

Example 2.5. Energy momentum tensor for a system of point-like particles.
To start with, consider a point-like particle moving along a trajectory $x^\mu = x^\mu(\tau)$ where τ is a parametrization of the curve. Such a system is described by the action (2.39) whose variation with respect to the metric can be calculated as follows

$$\begin{aligned}\delta S &= -m\int \left((g_{\mu\nu} + \delta g_{\mu\nu})dx^\mu dx^\nu\right)^{\frac{1}{2}} + m\int \left(g_{\mu\nu}dx^\mu dx^\nu\right)^{\frac{1}{2}}\\ &= -m\int \frac{1}{2}\frac{\delta g_{\mu\nu}dx^\mu dx^\nu}{\sqrt{g_{\mu\nu}dx^\mu dx^\nu}}\\ &= -m\int \frac{1}{2}\delta g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}d\tau\\ &= -m\int \frac{1}{2}(-g_{\alpha\mu}g_{\beta\nu}\delta g^{\alpha\beta})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}d\tau\end{aligned} \tag{2.143}$$

where from the first to second line we used $\sqrt{x+\delta x} \approx \sqrt{x} + \frac{\delta x}{2\sqrt{x}}$; from the second to third line we used $d\tau = \sqrt{g_{\mu\nu}dx^\mu dx^\nu}$; from the third to fourth line we used the identity (2.120). To bring the above expression to a functional derivative form, we include a 4-dimensional spacetime integral d^4x at the expense of slipping in a Dirac delta function $\delta^4(x^\mu - x^\mu(\tau))$ which makes sure that the particle follows the trajectory $x^\mu(\tau)$

$$\begin{aligned}\delta S &= \int d\tau \int d^4x\sqrt{-g}\frac{1}{\sqrt{-g}}\frac{m}{2}(g_{\alpha\mu}g_{\beta\nu}\delta g^{\alpha\beta})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta^4(x^\mu - x^\mu(\tau))\\ &= \int d^4x\frac{1}{2}\sqrt{-g}\delta g^{\alpha\beta}\left(\int d\tau\frac{m}{\sqrt{-g}}g_{\alpha\mu}g_{\beta\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta^4(x^\mu - x^\mu(\tau))\right)\end{aligned} \tag{2.144}$$

from which we can read the energy-momentum tensor for a point-like particle

$$T_{\mu\nu} = \frac{m}{\sqrt{-g}}\int d\tau\, g_{\mu\alpha}g_{\nu\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\delta^4(x^\mu - x^\mu(\tau)). \tag{2.145}$$

Due to the presence of the δ-function and $\sqrt{-g}$, one may question whether the energy-momentum tensor we just found is really a tensor. From the identity $1 = \int d^4x\delta^4(x^\mu - x^\mu(\tau))$, we can deduce that $\delta^4(x^\mu - x^\mu(\tau))$ transforms in a way that is opposite to how d^4x transforms

$$\delta^4\left(x^\mu - x^\mu(\tau)\right) \to \left(\det\frac{\partial x^\mu}{\partial x'^\alpha}\right)^{-1}\delta^4\left(x'^\alpha - x'^\alpha(\tau)\right). \tag{2.146}$$

Furthermore, since the determinant of the metric $\det g_{\mu\nu}$ transforms as

$$\det g_{\mu\nu} \to \det \left(\frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} \cdot g'_{\mu\nu} \right) = \left(\det \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} \right) \cdot \det g'_{\mu\nu} \tag{2.147}$$

$\sqrt{-g}$ transforms as

$$\sqrt{-g} \to \det \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \det \sqrt{-g'}. \tag{2.148}$$

Therefore the combination $\delta^4 \left(x'^{\alpha} - x'^{\alpha}(\tau)\right) / \sqrt{-g}$ is invariant under coordinate transformations and $T_{\mu\nu}$ indeed transforms as a tensor.

2.10.4 Physical meanings of the components of $T_{\mu\nu}$

The physical meanings of the components of the energy-momentum tensor $T_{\mu\nu}$ can be seen clearly in the case of a point-like particle. In a flat space time, we have $\sqrt{-g} = 1$ and $g_{\mu\nu} = \eta_{\mu\nu}$. Consequently, the energy momentum tensor (2.145) of a point-like particle becomes

$$T_{\mu\nu} = \frac{m}{\sqrt{1}} \int d\tau \, \eta_{\mu\alpha}\eta_{\nu\beta} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau} \delta^4(x^{\mu} - x^{\mu}(\tau)) = m \int d\tau \, \frac{dx_{\mu}}{d\tau} \frac{dx_{\nu}}{d\tau} \delta^4(x^{\mu} - x^{\mu}(\tau)). \tag{2.149}$$

Substituting the definition of 4-momentum $p_{\mu} = m\frac{dx_{\mu}}{d\tau}$, energy $E = \gamma m$ and proper time $d\tau = dx^0\sqrt{1-v^2} = \frac{dx^0}{\gamma}$, and decomposing the δ-function, we get

$$T_{\mu\nu} = \int dx^0 \frac{p_{\mu}p_{\nu}}{E} \delta^3 \left(x^i - x^i(\tau)\right) \delta \left(x^0 - x^0(\tau)\right). \tag{2.150}$$

After performing the integration, we find

$$T_{\mu\nu} = \frac{p_{\mu}p_{\nu}}{E} \delta^3(x^i - x^i(\tau)) \tag{2.151}$$

from which we can read off that

- $T_{00} = E\delta^3(x^i - x^i(\tau))$, i.e. T_{00} represents the energy density.
- $T_{0i} = p_i\delta^3(x^i - x^i(\tau))$, i.e. T_{0i} represents the momentum density.
- $T_{ij} = \frac{p_i p_j}{E}\delta^3(x^i - x^i(\tau))$, i.e. the flux of momentum p^i across the surface of constant x^i.

In general, the components of $T_{\mu\nu}$ can be understood as the flux of p^{μ} across a surface of constant x^{ν}. For example, the energy density T_{00} is the flux of energy across the surface of constant time. The total energy E and the total momentum P_i can therefore be obtained by integrating T_{00} and T_{0i} respectively over all space:

$$E = \int d^3x T_{00}, \qquad P_i = \int d^3x T_{0i}. \tag{2.152}$$

The conservation of the total energy of a system follows from integrating the local conservation of energy $T_{\mu\nu}{}^{;\nu} = T_{\mu\nu}{}^{,\nu} = 0$ (flat spacetime) for $\mu = 0$

$$0 = \frac{d}{dt}\int T_{00} d^3x - \int d^3x \frac{\partial T_{0i}}{\partial x^i} = \frac{dE}{dt} - \int d^3x \nabla \cdot \mathbf{P} = \frac{dE}{dt} - \underbrace{\int \mathbf{dS} \cdot \mathbf{P}}_{=0} = \frac{dE}{dt}. \tag{2.153}$$

To understand the meaning of T_{ij} better, consider the following example.

Example 2.6. Energy-momentum tensor of an ideal gas.
Consider a gas of non-interacting point-like particles in a flat spacetime. To describe such a system, we define the distribution function $f(\mathbf{x}, \mathbf{p})$, which tells us the number of particles in a volume dV around the point $\mathbf{x}$ and with momentum spread d^3p around the momentum $\mathbf{p}$. The distribution function $f(\mathbf{x}, \mathbf{p})$ is related to the number density $n(\mathbf{x})$ as

$$n(\mathbf{x}) = \int d^3\mathbf{p} f(\mathbf{x}, \mathbf{p}). \tag{2.154}$$

If the momentum distribution of the particles is isotropic, we can write the distribution function as

$$f(\mathbf{x}, \mathbf{p}) = n(\mathbf{x}) f(p) \tag{2.155}$$

with $p = |\mathbf{p}|$ and $\int f(p) d^3p = 1$. The energy-momentum tensor of the ideal gas can be computed by summing up the energy-momentum tensor (2.151) of individual particles we obtained earlier

$$T_{\mu\nu} = n(\mathbf{x}) \int d^3p \; f(p) \frac{p_\mu p_\nu}{E}. \tag{2.156}$$

We can easily see that

- $T_{00} = n(\mathbf{x}) \int d^3p \; f(p) E(p)$
- $T_{0i} = n(\mathbf{x}) \int d^3p \; f(p) p_i = 0$ since this integral is odd for $d^3p \to -d^3p'$.

To simplify our discussion, let us assume that the ideal gas is non-relativistic and in thermal equilibrium so that the $f(p)$ obeys the Maxwell-Boltzmann distribution

$$f(p) = C \exp\left(-\frac{p^2}{2mT}\right). \tag{2.157}$$

With this assumption T_{00} can be computed explicitly, yielding

$$T_{00} = n(\mathbf{x}) m + n(\mathbf{x}) 3T = \rho(\mathbf{x}) \tag{2.158}$$

i.e. the energy density at point $\mathbf{x}$. It requires a bit more work to obtain T_{ij}. First, notice that due to the isotropy assumptions we can write

$$\langle p_i p_j \rangle = \left\langle \left(p^i\right)^2 \right\rangle \eta_{ij} = \frac{1}{3} p^2 \eta_{ij}. \tag{2.159}$$

Then, we can compute T_{ij} as follows

$$T_{ij} = n(\mathbf{x}) \int d^3p \; f(p) \frac{p_i p_j}{E} = \frac{1}{3} n(\mathbf{x}) \eta_{ij} C \int d^3p \exp\left(-\frac{p^2}{2mT}\right) \frac{p^2}{m}. \tag{2.160}$$

The integral can be computed exactly and when the dust has settled we find that

$$T_{ij} = -n(x) \eta_{ij} 2T = P \eta_{ij} \tag{2.161}$$

where $P = 2Tn$ is the pressure (the force the particle bombardments would exert on a perfectly reflecting unit surface) of the ideal gas. Therefore, we conclude that the T_{ij} components of the energy-momentum tensor represent some sort of the pressure.

2.10.5 Newtonian limit of Einstein equations

One test of the success of General Relativity as a theory of gravity is that it must reproduce the results of Newtonian gravity in the limit where: the cosmological constant is negligible $\lambda = 0$, masses are "small" (to be clarified), and velocities are small $v \ll 1$. When we set $\lambda = 0$, the Einstein equations (2.133) reduce to

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}. \tag{2.162}$$

Taking the trace of both sides, we find

$$8\pi G T^{\mu}{}_{\mu} = 8\pi G T_{\mu\nu} g^{\mu\nu} = \underbrace{g^{\mu\nu} R_{\mu\nu}}_{=R} - \frac{1}{2} \underbrace{g^{\mu\nu} g_{\mu\nu}}_{=4} R = -R.$$

Thus, we can rewrite the Einstein equations as

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \tag{2.163}$$

with $T = T^{\mu}{}_{\mu}$. Recall that in the case of a point-like particle at rest we have $T_{00} = m\delta^3(x)$ and $T_{ij} = T_{0i} = 0$. In that case, the Einstein equation for $\mu = \nu = 0$ becomes

$$R_{00} = 8\pi G \left(T_{00} - \frac{1}{2} \underbrace{g_{00}}_{\approx 1} T \right) = 8\pi G \left(m\delta^3(x) - \frac{1}{2} T \right). \tag{2.164}$$

Since $T = T^{\mu}{}_{\mu} = g^{\mu\nu} T_{\nu\mu} = g^{00} T_{00} = T_{00}$, the above simplifies to

$$R_{00} = 4\pi G m \delta^3(x). \tag{2.165}$$

Now, R_{00} can also be computed using (2.69)

$$R_{00} = \frac{\partial \Gamma^{\lambda}_{00}}{\partial x^{\lambda}} - \cancel{\frac{\partial \Gamma^{\lambda}_{0\lambda}}{\partial x^0}} - \cancel{(\cdots)} \approx \Delta\varphi \tag{2.166}$$

where we have used

$$\Gamma^{i}_{00} = \frac{1}{2} \frac{\partial \delta g_{00}}{\partial x^i} \quad \overset{\delta g_{00} \approx 2\varphi}{\Longrightarrow} \quad \Gamma^{i}_{00} = \frac{\partial \varphi}{\partial x^i} \tag{2.167}$$

i.e. the only non-zero Christoffel symbols. Equating the newly obtained R_{00} to (2.165), we obtain

$$\Delta\varphi = 4\pi G m \delta^3(x) \tag{2.168}$$

which is exactly the Newton equation of gravity for a point particle, as expected.

2.10.6 Other possible terms in the action describing gravity

Setting aside the cosmological constant λ and dimensionless factors, the Einstein-Hilbert action reads

$$S_E \sim \int d^4x \sqrt{-g} G_N^{-1} R. \tag{2.169}$$

As a modification to the above action, one could conceive terms like R^2 or $R_{\mu\nu}R^{\mu\nu}$. We are going to show now that these terms are negligible in most cases of interest. Since the action has no mass dimension (GeV^0) and the dimension of the volume element d^4x is GeV^{-4}, each Lagrangian term must have the dimension of GeV^4. The R term of the Einstein-Hilbert action, for example, is made up of an $R \sim \partial^2 g_{\mu\nu}/\partial x^2$ whose dimension is GeV^2 and a $G_N^{-1} = M_P^2$ (where M_P is the Planck mass) whose dimension is GeV^2 so that they together make GeV^4. Any other term we wish to add to the Einstein-Hilbert action which involves some scalar function of $g_{\mu\nu}$ must be multiplied with an appropriate power of the Planck mass M_P to get the dimension right. For instance, consider the following two terms

$$\mathcal{L} \sim \underbrace{M_P^2 R}_{\sim M_P^2 (\Delta x)^{-2}} + \underbrace{R^2}_{\sim (\Delta x)^{-4}} \tag{2.170}$$

where Δx is the typical length scale of the system of interest. When we take the ratio of the two terms

$$\frac{R^2}{M_P^2 R} \sim \frac{(\Delta x)^{-4}}{M_P^2(\Delta x)^{-2}} \sim \frac{M_P^{-2}}{\Delta x^2} \tag{2.171}$$

we see that the R^2 term is negligible as long as the length scale of interest Δx is much larger than the Planck length $M_P^{-1} \sim 10^{-33}$ cm. Similar arguments apply to other higher-order terms that we can imagine adding to the Einstein-Hilbert action.

3 TESTS OF GENERAL RELATIVITY

3.1 GENERAL STATIC ISOTROPIC METRIC

In many cases of interest, the spacetime is approximately static and isotropic. It is therefore instructive to figure out the most general static and isotropic metric. Specifically, static means that the metric is time-independent and isotropic means that the metric is invariant under rotations. To ensure the former is satisfied each component of the metric must not depend on the time coordinate t and to ensure the latter we build the metric solely out of rotational invariants such as $\mathbf{dx}^2, \mathbf{x}\cdot\mathbf{dx}$ and $\mathbf{x}^2$. The most general metric with the aforementioned properties has the form

$$ds^2 = F(r)dt^2 - 2E(r)dt(\mathbf{x}\cdot\mathbf{dx}) - D(r)(\mathbf{x}\cdot\mathbf{dx})^2 - C(r)\mathbf{dx}^2 \tag{3.1}$$

where $F(r), E(r), D(r), C(r)$ are arbitrary functions of $r = \sqrt{\mathbf{x}\cdot\mathbf{x}}$. Let us now take several steps to simplify the above expression. First, we go to the spherical coordinates

$$\begin{cases} x^1 &= r\sin\theta\cos\phi \\ x^2 &= r\sin\theta\sin\phi \\ x^3 &= r\cos\theta. \end{cases} \tag{3.2}$$

Since

$$\mathbf{x}\cdot\mathbf{dx} = \frac{1}{2}d(\underbrace{x^2}_{=r^2}) = \frac{1}{2}d(r^2) = rdr \tag{3.3}$$

$$\mathbf{dx}^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \tag{3.4}$$

the metric can be written as

$$ds^2 = F(r)dt^2 - 2rE(r)drdt - r^2D(r)dr^2 - C(r)\left(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right) \tag{3.5}$$

We can remove the cross-term $drdt$ by shifting the time coordinate appropriately $t' = t + \Phi(r)$ while keeping the remaining coordinates unmodified $r' = r, \phi' = \phi, \theta' = \theta$. The time differentials are related as

$$dt = d(t' - \Phi(r)) = dt' - \frac{\partial\Phi(r')}{\partial r'}dr' \tag{3.6}$$

and so the new metric becomes

$$\begin{aligned} ds^2 =& F(r')\left(dt' - \frac{\partial\Phi}{\partial r'}dr'\right)^2 - 2E(r')r'dr'\left(dt' - \frac{\partial\Phi}{\partial r'}dr'\right) \\ &- r'^2D(r')dr'^2 - C(r')\left(dr'^2 + r'^2d\theta'^2 + r'^2\sin^2\theta' d\phi'^2\right) \end{aligned} \tag{3.7}$$

The $drdt$ term vanishes if we choose $\Phi(r)$ such that

$$\frac{d\Phi}{dr'} = -\frac{r'E(r')}{F(r')}. \tag{3.8}$$

Making another coordinate transformation $C(r')r'^2 = r''^2$ while keeping the rest of the coordinates unmodified, we arrive at the "standard form" of static isotropic metric

$$\boxed{ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)} \tag{3.9}$$

where we have gotten rid of all the primes and $A(r), B(r)$ are some combinations of the functions of r introduced earlier.

For future convenience, let us derive some useful formulas related to the static isotropic metric. We can read from (3.9) the non-vanishing components of the metric

$$g_{rr} = -A(r), \qquad g_{tt} = B(r), \qquad g_{\theta\theta} = -r^2, \qquad g_{\phi\phi} = -r^2\sin^2\theta$$

and its inverse

$$g^{rr} = -\frac{1}{A(r)}, \qquad g^{tt} = \frac{1}{B(r)}, \qquad g^{\theta\theta} = -\frac{1}{r^2}, \qquad g^{\phi\phi} = -\frac{1}{r^2\sin^2\theta}.$$

From these components, we can calculate the determinant

$$g = \det g_{\mu\nu} = -r^4\sin^2\theta A(r)B(r) \tag{3.10}$$

and

$$\sqrt{-g} = r^2\sin\theta\sqrt{A(r)B(r)}. \tag{3.11}$$

Using (1.106), we obtain the non-vanishing Christoffel symbols

$$\begin{aligned}
\Gamma^r_{rr} &= \frac{A'}{2A} & \Gamma^r_{\theta\theta} &= -\frac{r}{A} \\
\Gamma^r_{\phi\phi} &= -\frac{r\sin^2\theta}{A} & \Gamma^r_{tt} &= \frac{B'}{2A} \\
\Gamma^\theta_{r\theta} &= \frac{1}{r} & \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma^\phi_{\phi r} &= \frac{1}{r} & \Gamma^\phi_{\phi\theta} &= \cot\theta \\
\Gamma^t_{tr} &= \frac{B'}{2B}
\end{aligned} \tag{3.12}$$

from which we can compute the non-zero components of the Ricci tensor

$$\begin{aligned}
R_{rr} &= -\left[\frac{B''}{2B} - \frac{1}{4}\left(\frac{B'}{B}\right)\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\frac{A'}{A}\right] \\
R_{\theta\theta} &= -\left[-1 + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A}\right] \\
R_{\phi\phi} &= \sin^2\theta R_{\theta\theta} \\
R_{tt} &= -\left[-\frac{B''}{2A} + \frac{1}{4}\left(\frac{B'}{A}\right)\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\left(\frac{B'}{A}\right)\right].
\end{aligned} \tag{3.13}$$

3.2 SCHWARZSCHILD METRIC

In this section, we consider a special case of the static isotropic metric (3.9) in a spacetime with a spherically-symmetric mass distribution concentrated around the origin ($r = 0$). In particular, we are interested in spacetime regions sufficiently far from the origin where matter is, for all intent and purposes, absent. The metric in those vacuum regions has a special form and is known as the *Schwarzschild metric*. In Newtonian gravity, such a system is described by the Laplace equation $\nabla^2\varphi = 0$ and the solution is $\varphi = C/r$ which is valid everywhere except near the origin where matter is present. Let us now derive the General Relativistic equivalent of it. The Einstein equation (2.133) in the absence of matter (2.133) reduces to

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \tag{3.14}$$

Taking its trace

$$\underbrace{g^{\mu\nu}R_{\mu\nu}}_{=R} - \frac{1}{2}R\underbrace{g^{\mu\nu}g_{\mu\nu}}_{=4} = R(1-2) = -R = 0$$

we find that the Ricci scalar R vanishes, which, in turn, means that the Ricci tensor $R_{\mu\nu}$ also vanishes, owing to the Einstein equation (3.14). Since the Ricci tensor for a static isotropic spacetime can be written in terms of two functions, A and B, as found in (3.13), the condition $R_{\mu\nu} = 0$ provides us with 4 equations for 2 functions. There is, in fact, a redundancy. The $R_{\theta\theta} = 0$ equation and the $R_{\phi\phi} = 0$ equation give the same constraint. Furthermore, as we are going to show now, the remaining 3 equations can be reduced to 2. Consider the following combination of the $R_{rr} = 0$ and $R_{tt} = 0$ equations

$$\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = \frac{1}{r}\left(\frac{A'}{A^2} + \frac{B'}{AB}\right) = 0 \quad \Longrightarrow \quad \frac{A'}{A} + \frac{B'}{B} = 0. \tag{3.15}$$

When written in a different form

$$A'B + B'A = (AB)' = 0$$

it clearly implies that

$$AB = \text{constant}. \tag{3.16}$$

Far away from the origin at $r \to \infty$, we expect the spacetime to be flat and so $A \to 1$ and $B \to 1$. This implies that the constant in the equation above must be equal to 1

$$AB = 1. \tag{3.17}$$

This equation allows us to rewrite the $R_{\theta\theta} = 0$ and $R_{rr} = 0$ equations as

$$\begin{cases} R_{\theta\theta} & = 1 - rB' - B = 0 \\ R_{rr} & = -\frac{B''}{2B} - \frac{B'}{rB} = -\frac{B'+rB''+B'}{2rB} = \frac{R'_{\theta\theta}}{2rB} = 0 \end{cases} \tag{3.18}$$

which demonstrates that these equations are equivalent in terms of the constraints they give. It is therefore sufficient to consider only $R_{\theta\theta} = 0$. It can be written as

$$(rB)' = 1 \tag{3.19}$$

which can be solved easily

$$rB = r + \text{constant} \quad \Longrightarrow \quad B = 1 + \frac{C}{r}. \tag{3.20}$$

To determine the constant C, we use the weak-field limit correspondence $B \equiv g_{00} \approx 1 + 2\varphi$, where in this case $\varphi = -GM/r$. Thus, we find

$$B(r) = 1 - \frac{2MG}{r} \tag{3.21}$$

and by (3.17)

$$A(r) = \frac{1}{1 - 2GM/r}. \tag{3.22}$$

All in all, the metric is given by

$$\boxed{ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \frac{dr^2}{1 - 2GM/r} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2.} \tag{3.23}$$

This metric is known as the *Schwarzschild metric*. The combination $r_g = 2GM$ is often referred to as the *gravitational radius* or *Schwarzschild radius*. Something interesting happens at $r = r_g$. At that point $A = \infty$ and $B = 0$. We will come back to this issue later. Furthermore, in the $r < r_g$ region both A and B change sign. As a result, the roles of the radial coordinate r and the temporal coordinate t are interchanged; r behaves like a temporal coordinate and t behaves like a spatial coordinate.

3.3 MOTIONS IN SCHWARZSCHILD SPACETIME

3.3.1 Equations of motion

Now that we have computed the Schwarzchild metric

$$ds^2 = \left(1 - \frac{2MG}{r}\right) dt^2 - \frac{1}{1 - \frac{2MG}{r}} dr^2 - r^2 d\Omega \tag{3.24}$$

with $d\Omega = d\theta^2 + \sin^2\theta d\phi^2$ we can proceed to study the motion of a particle in a Schwarzschild spacetime. When we want to describe the motion of a particle in a curved space, the first equation that comes to mind is the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0. \tag{3.25}$$

Instead of using the geodesic equation, we take this opportunity to show the power of the action principle, which makes the computations simpler. In fact, we will also show in passing that the Christoffel symbols can be obtained relatively easily by comparing the equation of motion obtained from the action principle with the geodesic equation. Consider the action

$$S = -\frac{1}{2}m\int d\tau\, g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{3.26}$$

whose extrema describe the motion of a free particle. Knowing the Schwarzschild metric, we can write the above action more explicitly

$$S = -\frac{1}{2}m\int d\tau \left[B(r)\left(\frac{dt}{d\tau}\right)^2 - A(r)\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\theta}{d\tau}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]. \tag{3.27}$$

Note that we do not include the Einstein-Hilbert action in the action we are considering. This amounts to ignoring the backreaction of the particle to the spacetime. In other words, we treat the metric as a fixed background.

Extremizing the action over variations in t, ϕ, θ, and r, we obtain the following equations of motion

$$t: 0 = m\frac{d}{d\tau}\left(B\frac{dt}{d\tau}\right)$$

$$\phi: 0 = m\frac{d}{d\tau}\left(r^2\sin^2\theta\frac{d\phi}{d\tau}\right)$$

$$\theta: 0 = m\frac{d}{d\tau}\left(r^2\frac{d\theta}{d\tau}\right) - r^2\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2$$

$$r: 0 = \frac{d}{d\tau}\left(A\frac{dr}{d\tau}\right) + \frac{1}{2}\frac{\partial B}{\partial r}\left(\frac{dt}{d\tau}\right) - \frac{1}{2}\frac{dA}{dr}\left(\frac{dr}{d\tau}\right)^2 - \frac{2r\sin^2\theta}{2}\left(\frac{d\phi}{d\tau}\right)^2 - r\left(\frac{d\theta}{d\tau}\right)^2$$

respectively. By deriving these equations, we have indirectly calculated the non-zero Christoffel symbols. Take the first equation, for example. We can rewrite it as

$$\frac{d^2t}{d\tau^2} + \frac{B'}{B}\frac{dr}{d\tau}\frac{dt}{d\tau} = 0. \tag{3.28}$$

Based on the form of the second term, we can deduce that it corresponds to the geodesic equation (3.25) for $\mu = t$ and it follows that

$$\Gamma^t_{rt} = \frac{B'}{2B}. \tag{3.29}$$

Taking advantage of general covariance, without loss of generality we can go to a coordinate system where, say, $\theta = \pi/2$. According to the equation of motion obtained from varying θ, we have $d\theta/dt = 0$ in this coordinate system. Therefore, in general the particle moves on a constant θ plane, which in this case is chosen to be the $\theta = \pi/2$ plane. The equations of motion obtained from varying t and varying ϕ then reduce to

$$m\frac{d}{d\tau}\left(B\frac{dt}{d\tau}\right) = 0$$
$$m\frac{d}{d\tau}\left(r^2\frac{d\phi}{d\tau}\right) = 0.$$

Integrating them gives us two integrals of motion E and J

$$E = mB\frac{dt}{d\tau}$$
$$J = mr^2\frac{d\phi}{d\tau}. \tag{3.30}$$

E can be interpreted as energy and J can be interpreted as angular momentum. The expression for J agrees with the non-relativistic formula for angular momentum, $\mathbf{r} \times \mathbf{p}$, and in the $r \to \infty$ limit the expression for E simplifies to

$$mB\frac{dt}{d\tau} \overset{B\approx 1}{=} m\frac{dt}{d\tau} = \frac{m}{\sqrt{1-v^2}}$$

which is the relativistic formula for energy.

Now, consider the following identity that follows from the definition of proper time $d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu$

$$1 = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}. \tag{3.31}$$

Substituting in the Schwarzschild metric, we get

$$B\left(\frac{dt}{d\tau}\right)^2 - A\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\phi}{d\tau}\right)^2 = 1. \tag{3.32}$$

Using the conserved quantities (3.30) we recently found, we can rewrite it as

$$B\left(\frac{E}{mB}\right)^2 - A\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{J}{mr^2}\right)^2 = 1 \tag{3.33}$$

or

$$A\left(\frac{dr}{d\tau}\right)^2 + \frac{J^2}{m^2r^2} - \frac{E^2}{m^2B} = -1. \tag{3.34}$$

Notice that the derivative of this equation coincides with the equation of motion from the variation of r, which is not a surprise as the 4 equations of motion are enough to determine the dynamics of the 4 coordinates t, ϕ, θ, and r. Adding the proper time identity to the set of equations would only make the resulting set redundant.

To recap, the motion of a particle in a Schwarzschild spacetime is dictated by the following equations

$$\theta = \pi/2$$
$$E = mB\frac{dt}{d\tau}$$
$$J = r^2\frac{d\phi}{d\tau}$$
$$-1 = A\left(\frac{dr}{d\tau}\right)^2 + \frac{J^2}{m^2r^2} - \frac{E^2}{m^2B}.$$

The above equations, however, only apply to massive particles. This is because they involve $d\tau$'s which are zero for massless particles. In order to describe massless particles, we need to parameterize the time with a different parameter than τ. We can choose the parameter to be the coordinate time t

$$\begin{aligned} \theta &= \pi/2 \\ \frac{J}{E} &= \frac{1}{B} r^2 \frac{d\phi}{dt} \\ 0 &= A\frac{E^2}{B^2}\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{r^2} - \frac{E^2}{B}. \end{aligned} \tag{3.35}$$

3.3.2 Open orbits: light bending

The trajectory of a particle in an isotropic static spacetime can be described by the function $r(\phi)$ or its inverse $\phi(r)$. Let us compute such a function. We can rewrite (3.34) in terms of r and ϕ only by making use of the angular momentum conservation equation $J = mr^2 d\phi/d\tau$

$$\frac{E^2}{B} - \frac{AJ^2}{r^4}\left(\frac{dr}{d\phi}\right)^2 - \frac{J^2}{r^2} = m^2. \tag{3.36}$$

All that is left to be done is solving this equation. After rearranging it to

$$\frac{A}{r^4} dr^2 + d\phi^2 \left(\frac{1}{r^2} + \frac{m^2}{J^2} - \frac{E^2}{BJ^2}\right) = 0, \tag{3.37}$$

moving one of the terms to the right hand side, and integrating, we get

$$\boxed{\phi(r) = \pm \int dr \frac{\sqrt{A}}{r^2\left(\frac{E^2}{BJ^2} - \frac{m^2}{J^2} - \frac{1}{r^2}\right)^{\frac{1}{2}}}.} \tag{3.38}$$

This equation is valid not only for Schwarzschild spacetimes but also for any static and isotropic spacetime. If the particle in question is massless, we can simply set $m = 0$.

Consider a particle of mass m moving towards the Sun from far away with an approximately constant velocity v. We are interested in calculating to what extent the particle's trajectory is bent by the gravitational pull of the Sun. While the particle is still far away, say at the coordinates r and $\phi \ll 1$, its impact parameter b is given by [1]

$$b \approx r \sin\phi \approx r\phi \tag{3.39}$$

and its radial velocity is given by

$$-v \approx \frac{d(v\cos\phi)}{dt} \approx \frac{dr}{dt}. \tag{3.40}$$

Thus, the particle's angular momentum and energy are given by

$$J = \frac{mvb}{\sqrt{1-v^2}} \tag{3.41}$$

$$E = \frac{m}{\sqrt{1-v^2}}. \tag{3.42}$$

However, our calculations would be simpler had we expressed the angular momentum J in terms of r_0, the particle's closest distance to the Sun, instead of the impact parameter b. At the point of minimal distance, we have

$$\frac{dr}{d\phi}(r_0) = 0, \tag{3.43}$$

[1] If the gravitational pull is turned off, the particle's trajectory would be a straight line. The impact parameter b is the shortest distance from the origin (where the Sun is located) to this line.

and (3.37) becomes

$$\frac{E^2}{BJ^2} - \frac{m^2}{J^2} - \frac{1}{r_0^2} = 0 \tag{3.44}$$

Making use of (3.42), we obtain an expression of J in terms of r_0

$$J = \frac{r_0 m}{\sqrt{1-v^2}} \sqrt{\frac{1}{B(r_0)} - 1 + v^2}. \tag{3.45}$$

Now we want to plug this J into (3.38). Let us start by working out the denominator of the integral

$$\begin{aligned} \frac{E^2}{BJ^2} - \frac{m^2}{J^2} - \frac{1}{r^2} &= \frac{1}{J(r_0)^2}\left(\frac{m^2\gamma^2}{B(r)} - m^2\right) - \frac{1}{r^2} \\ &= \frac{1}{r_0^2 \cancel{m^2}\gamma^2\left(\frac{1}{B(r_0)} - 1 + v^2\right)}\left(\frac{\cancel{m^2}\gamma^2}{B(r)} - \cancel{m^2}\right) - \frac{1}{r^2} \\ &= \frac{1}{r_0^2}\left(\frac{\frac{1}{B(r)} - 1 + v^2}{\frac{1}{B(r_0)} - 1 + v^2}\right) - \frac{1}{r^2}. \end{aligned} \tag{3.46}$$

Hence, the full expression of (3.38) becomes

$$\phi(r) = \int_r^\infty \frac{\sqrt{A}dr}{r^2\left(\frac{1}{r_0^2}\frac{\frac{1}{B(r)}-1+v^2}{\frac{1}{B(r_0)}-1+v^2} - \frac{1}{r^2}\right)^{\frac{1}{2}}} \tag{3.47}$$

Notice that $\phi(r) \to 0$ when $r \to \infty$, as it should. To make our calculation simpler, let us suppose that the particle we are considering is massless so that we can set $v = 1$. The above equation then simplifies to

$$\phi(r) = \int_r^\infty \frac{\sqrt{A}dr}{r^2\left(\frac{1}{r_0^2}\frac{B(r_0)}{B(r)} - \frac{1}{r^2}\right)^{\frac{1}{2}}} = \int_r^\infty \frac{\sqrt{A}dr}{r\left(\frac{r^2}{r_0^2}\frac{B(r_0)}{B(r)} - 1\right)^{\frac{1}{2}}}. \tag{3.48}$$

As we can see in Figure 1, the angle of bending is given by

$$\Delta\phi = 2\phi(r_0) - \pi. \tag{3.49}$$

Let us now assume that $r_0 \gg r_g$ to further simplify our calculations. This allows us to make the following approximations

$$A(r) \approx 1 + 2\frac{MG}{r}$$
$$B \approx 1 - 2\frac{MG}{r}$$

In this approximation, we can compute $\phi(r_0)$, and therefore $\Delta\phi$ explicitly (see exercise 1 set 12). The result is

$$\Delta\phi = \frac{4MG}{r_0} + O\left(\frac{r_g}{r_0}\right)^2. \tag{3.50}$$

Note that at this level of accuracy, the impact parameter b and minimal distance r_0 coincide. If we plug in the following numbers: $M = M_\odot = 1{,}97 \times 10^{30}$ g, $r_0 = R_\odot = 6{,}95 \times 10^5$ km we get

$$\Delta\phi = 1.75 \text{ arcseconds} \tag{3.51}$$

which is extremely small. One way to observe this effect is by measuring the shifts in the apparent positions of stars when they are close to the sun, angularly speaking. Due to such shifts, some stars located behind the sun can be visible.

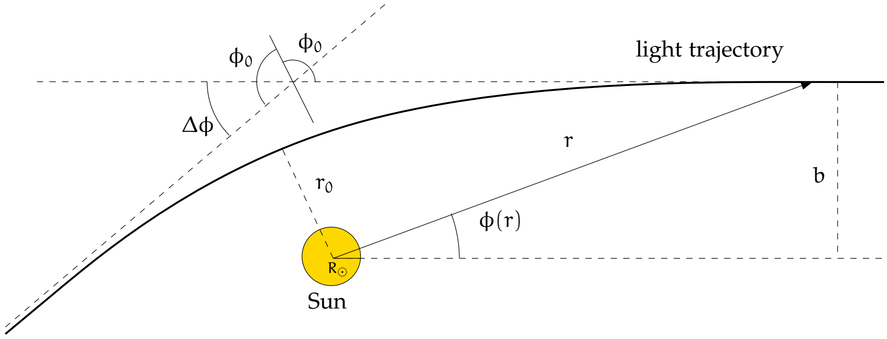

Figure 1: Light bending by the Sun.

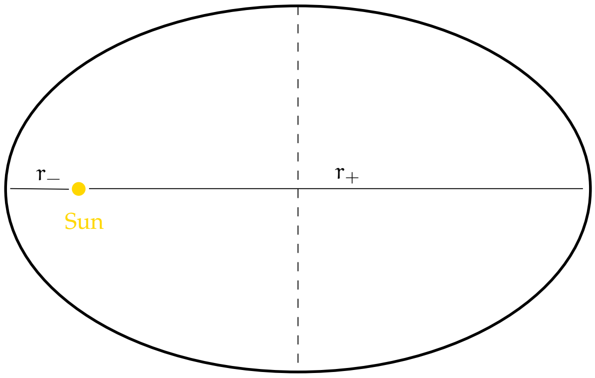

Figure 2: The orbit of a planet around the Sun.

3.3.3 Closed orbit: Precession of Mercury perihelium

Newtonian gravity predicts that planets orbiting around the Sun follow closed elliptical trajectories (see Figure 2). The minimum distance r_- (perihelium) and maximum distance r_+ (aphelium) of a planet to the Sun are given by the conditions

$$\frac{dr}{d\phi}(r_\pm) = 0. \tag{3.52}$$

The same conditions apply in GR, though the orbit is more complicated. At these extremum points, (3.34) reduce to

$$\frac{1}{r_\pm^2} + \frac{m^2}{J^2} - \frac{E^2}{B(r_\pm)}\frac{1}{J^2} = 0 \tag{3.53}$$

From these two equations, we can obtain the two integrals of motion J and E

$$\frac{m^2}{E^2} = \frac{1}{r_-^2 - r_+^2}\left(\frac{r_-^2}{B_-} - \frac{r_+^2}{B_+}\right), \qquad \frac{J^2}{E^2} = \left(\frac{1}{B_+} - \frac{1}{B_-}\right)\frac{r_-^2 r_+^2}{(r_-^2 - r_+^2)}. \tag{3.54}$$

One crucial thing that distinguishes general relativistic planetary orbits from Newtonian ones is that general relativistic orbits are typically not closed in the sense that the perihelium and aphelium of the orbit do not stay at the same angular coordinates, but instead they precess. If the orbit of a planet were to be closed, twice the angle between the perihelium and aphelium $2|\phi(r_+) - \phi(r_-)|$ must amount to exactly 2π. This is not the case when general relativistic effects are not negligible. They cause the angular position of the perihelium to shift by some amount

$$\Delta\phi = 2\,|\phi(r_+) - \phi(r_-)| - 2\pi. \tag{3.55}$$

in each revolution, where $\phi(r_+) - \phi(r_-)$ is given by

$$\phi(r_+) - \phi(r_-) = \int_{r_-}^{r_+} \frac{\sqrt{A}dr}{r^2\left(\frac{E^2}{BJ^2} - \frac{m^2}{J^2} - \frac{1}{r^2}\right)} \tag{3.56}$$

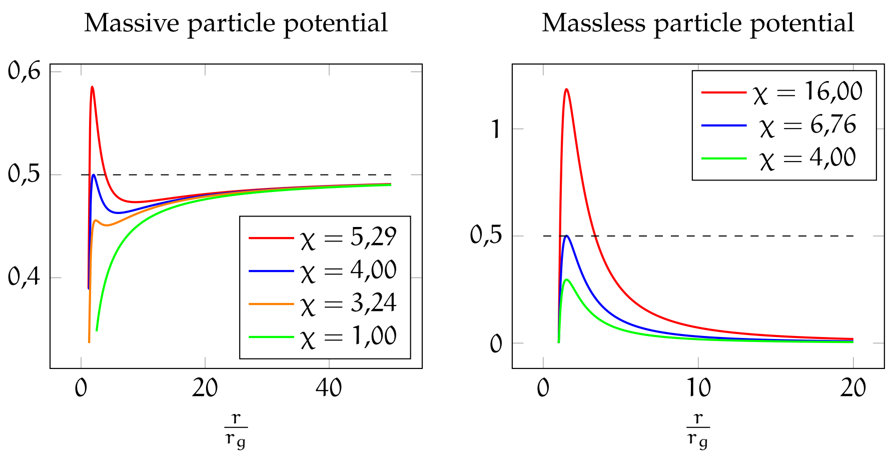

Figure 3: Effective potential for different values of $\chi = \frac{J}{mr_g}$ in the massless and massive particle case.

and E and J were found in (3.54). This integral can be computed exactly when $r_\pm \gg r_g$ (see exercise set 13). When the dust has settled, one finds

$$\Delta\phi \approx 6\pi \frac{M_\odot G}{L} \quad \text{where} \quad \frac{1}{L} = \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right). \tag{3.57}$$

Among the planets in our solar system, this effect is strongest for Mercury as it has the smallest L. For $L_{\text{Mercury}} = 55{,}4 \times 10^6$ km we find

$$\Delta\phi_{\text{Mercury}} = 0.1038 \text{ arcseconds} \tag{3.58}$$

for one revolution, which is a tiny shift (only 43 arcseconds in 100 years). Furthermore, this even harder to observe because it is not the sole effect that contributes to the $\Delta\phi$. For example, Newtonian gravity predicts that the gravitational pulls of the various planets in the solar system give a $\Delta\phi = 532$ arcseconds in 100 years, i.e. about 10 times larger than the general relativistic precession. On top of that, there are other effects that we have not taken into account here, e.g. the spherically asymmetry of the Sun, that may modify $\Delta\phi$ further. Despite all these difficulties, experimentalists have managed to identify and confirm the general relativistic contribution to $\Delta\phi$.

3.3.4 Effective potential and orbits

So far, in deriving various results, we have assumed that $r/r_g \gg 1$. In this section, we will not make such an assumption. The radial equation of motion (3.34) can be written in a form that eases physical interpretation

$$\underbrace{\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2}_{\text{"kinetic energy"}} + \underbrace{\frac{1}{2}\left(1 - \frac{r_g}{r}\right)\left(1 + \frac{J^2}{m^2 r_g^2}\frac{r_g^2}{r^2}\right)}_{\text{"potential energy"}} = \underbrace{\frac{1}{2}\frac{E^2}{m^2}}_{\text{"total energy"}}. \tag{3.59}$$

As indicated, we can interpret the above equation as the energy conservation of a point particle in one dimension moving under the influence of an *effective potential*

$$U = \frac{1}{2}\left(1 - \frac{r_g}{r}\right)\left(1 + \frac{J^2}{m^2 r_g^2}\frac{r_g^2}{r^2}\right) \tag{3.60}$$

with total energy total energy $E_{\text{tot}} = \frac{1}{2}\frac{E^2}{m^2}$.

The dimensionless combination of constants $\chi = \frac{J}{m r_g}$ determines the different possible shapes of the effective potential. See Figure 3 for representatives of different possible shapes of the effective potential. Let us focus on the massive particle case for the moment. Depending on the value of χ there can be 0 or 2 points where $dU(r)/dr = 0$ (turning points). When there are 2 turning points, the one with smaller r is a local maximum (where circular orbits are unstable) and the one with larger r is a local minimum (where circular orbits are stable). The turning points can be found by setting $dU(r)/dr = 0$

$$r = r_g \chi^2 \left[1 \pm \sqrt{1 - \frac{3}{\chi^2}}\right]. \tag{3.61}$$

Thus, there is no turning point if $\chi < \sqrt{3}$ and there are 2 turning points if $\chi > \sqrt{3}$. In particular, for the latter case, say when $\chi = 2$, there are several possible types of trajectories depending on the total energy of the particle, its starting point, and its initial velocity

- $U_{min} < \frac{1}{2}\frac{E^2}{m^2} < U_\infty$: closed orbits.
- $U_\infty < \frac{1}{2}\frac{E^2}{m^2} < U_{max}$: open orbit (scattering) if the particle starts outside the radius of U_{max} or falling into the center if it starts inside.
- $U_{max} < \frac{1}{2}\frac{E^2}{m^2}$: falling into center if the particle moves radially inward or going to infinity if it moves outward.

where U_{min}, U_{max}, and U_∞ are the local minimum, local maximum, and value at $r \to \infty$ of $U(r)$ respectively.

Now, we turn to massless particles. Starting from (3.35), we can switch to the parameterization $d\lambda = B dt$ to simplify things more. The radial equation of motion then becomes

$$\underbrace{\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}\left(1 - \frac{r_g}{r}\right)\frac{J^2}{E^2 r^2}}_{\text{potential energy}} = \underbrace{\frac{1}{2}}_{\text{total energy}}. \tag{3.62}$$

In this massless case, the total energy is fixed to be $1/2$ and the effective potential energy has always one maximum V_{max}. There are two possible types of motion

- $V_{max} < 1/2$: light deflection/scattering.
- $V_{max} > 1/2$: falling into the singularity.

3.4 BLACK HOLES

3.4.1 Falling radially into a black hole

Our aim here is to calculate the time it takes for a particle to fall into the point of singularity ($r = 0$) of a black hole. For simplicity, consider a massive particle moving radially inward (with zero $J = 0$) towards a black hole. Suppose that the particle is released with zero radial velocity $dr/dt = 0$ at $r = r_0$. Imposing these initial conditions on the radial equation of motion (3.34), we find

$$\frac{E^2}{m^2} = \left(1 - \frac{r_g}{r_0}\right) = 0$$

and so

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\cancel{1} - \frac{r_g}{r_0}\right) - \left(\cancel{1} - \frac{r_g}{r}\right) = r_g\left(\frac{1}{r} - \frac{1}{r_0}\right). \tag{3.63}$$

Taking the square root and rearranging

$$\frac{dr}{d\tau} = \pm\sqrt{r_g\left(\frac{1}{r} - \frac{1}{r_0}\right)} \quad \Longrightarrow \quad \frac{dr}{\sqrt{r_g}}\sqrt{\frac{rr_0}{r_0 - r}} = -d\tau. \tag{3.64}$$

Then, we perform a change of variables $\frac{r}{r_0} = z$

$$dz\sqrt{\frac{z}{1-z}} = -\frac{d\tau}{r_0}\sqrt{\frac{r_g}{r}}. \tag{3.65}$$

Let us first compute the amount of proper time τ_g it takes for the particle to reach $r = r_g$ starting from $r = r_0$

$$\tau_g = r_0\sqrt{\frac{r_0}{r_g}}\int_{\frac{r_g}{r_0}}^{1} \frac{dz\sqrt{z}}{\sqrt{1-z}}. \tag{3.66}$$

One can check that the result is finite. Moreover, the amount of proper time τ_0 it takes to reach the singularity $r = 0$ starting from $r = r_0$

$$\tau_g = r_0\sqrt{\frac{r_0}{r_g}}\int_{0}^{1} \frac{dz\sqrt{z}}{\sqrt{1-z}} = \frac{\pi}{2}r_0\sqrt{\frac{r_0}{r_g}} \tag{3.67}$$

is also finite.

Now, we repeat the same calculation from the perspective of an observer at rest at $r = r_0$ and assume that $r_0 \gg r_g$. The time interval dt as measured by the observer's clock is related to the proper time interval $d\tau$ of the falling particle by the energy conservation equation

$$E = mB\frac{dt}{d\tau}. \tag{3.68}$$

Thus, the falling time of the particle as measured by the observer's clock is

$$\int dt = \frac{E}{m}\int\frac{d\tau}{B} = \frac{E}{m}\int\frac{1}{B(r)}\frac{d\tau}{dr}dr = \frac{E}{m}\int_r^{r_0}\frac{r'}{r'-r_g}\sqrt{\frac{r'r_0}{(r_0 - r')r_g}}dr'. \tag{3.69}$$

Interestingly, the integral diverges in the limit $r \to r_g$. In other words, the observer will never see the particle crossing the $r = r_g$ horizon. To get an estimate of how quickly the time diverges, we can approximate all the r''s in the integral with r_g except for the one that is subtracted by r_g

$$\int dt \approx \frac{E}{m}\int_r^{r_0}\frac{r_g}{r'-r_g}\sqrt{\frac{\cancel{r_g}r_0}{(r_0 - r_g)\cancel{r_g}}} = \frac{E}{m}r_g\left[\log\frac{r_g}{r - r_g} + \text{const}\right]. \tag{3.70}$$

To recap, we found that it takes finite times to go from $r = r_0$ to $r = r_g$ and from $r = r_g$ to $r = 0$ from the falling particle's perspective, but infinite time to go from $r = r_0$ to $r = r_g$ from the perspective of an observer at rest at $r = r_0 \gg r_g$. How do we reconcile these two perspectives? To connect the two perspectives, suppose that the in-falling particle constantly emits electromagnetic wave of frequency ν_0 towards the observer at $r = r_0$. Recall that due to gravitational redshift a photon emitted with frequency ν_1 at a point x_1 is perceived as having a frequency ν_2 at a different point x_2

$$\frac{\nu_2}{\nu_1} = \sqrt{\frac{g_{00}(x_1)}{g_{00}(x_2)}}. \tag{3.71}$$

Let x_1 be the coordinate r of the particle and $x_2 = r_0 \gg r_g$ be the coordinate of the observer (at which point the spacetime is essentially flat, $g_{00}(x_2) \approx 1$. Hence, the frequency of the electromagnetic wave detected by the observer is given by

$$\nu_{obs} \equiv \nu_2 = \nu_1\sqrt{1 - \frac{r_g}{r}}. \tag{3.72}$$

As we can see, when $r \to r_g$ the frequency ν_{obs} approaches zero. Consequently, the particle appears frozen from the perspective of the observer.

Remark 3.1. Mechanism of black hole production.
Among other possible mechanisms, black holes may form from the gravitational collapse of a star, e.g. a neutron star. The physics of a collapsing star is extremely complicated. Apart from its dynamical and non-equilibrium nature, we also need to keep track of various effects, e.g. nuclear reactions, neutrinos, etc. Consequently, one often needs to make various assumptions such as zero pressure or spherical symmetry to get results.

Remark 3.2. The singular behaviour of the Schwarzschild metric at $r = r_g$ is just a coordinate artifact, nothing abrupt occurs at that point.
To justify this claim, we can construct all possible scalars from the metric $g_{\mu\nu}$ and check if any of them blows up at $r = r_g$. Since $R_{\mu\nu} = 0$ in a Schwarzschild spacetime, the consideration in Remark 2.9 tells us that there are only 4 such scalars. Here we list them without justification

$$C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}; \qquad \frac{1}{\sqrt{-g}}\epsilon^{\lambda\mu}{}_{\rho\sigma}C^{\rho\sigma\nu\kappa}C_{\lambda\mu\nu\kappa};$$

$$C_{\lambda\mu\nu\kappa}C^{\nu\kappa\rho\sigma}C_{\rho\sigma}{}^{\lambda\mu}; \qquad \frac{1}{\sqrt{-g}}C_{\lambda\mu\nu\kappa}C^{\nu\kappa\rho\sigma}\epsilon_{\rho\sigma}{}^{\alpha\beta}C_{\alpha\beta}{}^{\lambda\mu}$$

where $C_{\lambda\mu\nu\kappa}$ is the *Weyl tensor*, defined as

$$C_{\lambda\mu\nu\kappa} = \frac{R}{6}\left(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}\right) - \frac{1}{2}\left(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu}\right) - R_{\lambda\mu\nu\kappa} \tag{3.73}$$

with the property $C^{\lambda}{}_{\mu\nu\lambda} = 0$. Since the 4 scalars that we have listed are coordinate independent, it is presumably simplest to compute them in the local Minkowskian coordinate system. One can check that none of them is singular at $r = r_g$. On the other hand, these scalars blow up $r = 0$, signifying that it is a true singularity.

The above discussion implies that we can construct a coordinate system where the event horizon is just an ordinary, non-singular point. Indeed, there are many such coordinates. One of them are the so-called Krustal-Szekeres coordinates

$$u = \left(\frac{r}{r_g} - 1\right)^{\frac{1}{2}} e^{\frac{r}{2r_g}} \cosh\frac{t}{2r_g}$$

$$v = \left(\frac{r}{r_g} - 1\right)^{\frac{1}{2}} e^{\frac{r}{2r_g}} \sinh\frac{t}{2r_g}$$

for $r > r_g$ and

$$u = \left(1 - \frac{r}{r_g}\right)^{\frac{1}{2}} e^{\frac{r}{2r_g}} \sinh\frac{t}{2r_g}$$

$$v = \left(1 - \frac{r}{r_g}\right)^{\frac{1}{2}} e^{\frac{r}{2r_g}} \cosh\frac{t}{2r_g}$$

for $r < r_g$. The metric in this coordinate system is

$$ds^2 = \frac{4r_g^3}{r}e^{-r/r_g}\left(dv^2 - du^2\right) - r^2 d\Omega^2 \tag{3.74}$$

where r is understood as a function of u and v. Clearly, the metric is not singular at $r = r_g$, as promised.

3.4.2 Event horizon

One of many conclusions we can draw from the discussion in Section 3.3.4 is that for all cases, massive and massless included, if a particle reaches $r < r_g$, it will inevitably fall into the singularity at $r = 0$ (in Newtonian gravity $r_g = 0$ and so there is no case where the particle falls into the singularity). To put it simply, everything that comes inside the region $r < r_g$ does not come out. Since we cannot see what is inside the region $r < r_g$ (nothing can come out for us to see!), the surface $r = r_g$ is also known as the *event horizon*. A hole from which nothing can escape would presumably look black (setting aside Hawking radiation), hence the name "black hole."

The presence of the event horizon is best understood with the concept of escape velocity (the minimum velocity a particle needs to have in order to escape to $r \to \infty$). For simplicity, we again assume that the motion of the particle is purely radial. Take a look at the radial equation of motion

$$\frac{dr}{d\tau} = +\sqrt{\frac{E^2}{m^2} - \left(1 - \frac{r_g}{r}\right)}. \tag{3.75}$$

If the radial velocity $dr/d\tau$ at a particular radius r is to be the escape velocity, the energy E corresponding to it must be equal to the energy of a particle with velocity approaching zero at $r \to \infty$, namely $E = m$. Therefore, the escape velocity can be found by setting $E = m$ in the above equation

$$\left(\frac{dr}{d\tau}\right)_{\text{esc}} = \sqrt{\frac{r_g}{r}}. \tag{3.76}$$

As we can see $(dr/d\tau)_{\text{esc}}$ approaches the speed of light as $r \to r_g$ and even exceeds the speed or light when $r < r_g$, thus explaining why a particle is bound to fall into the singularity once it reaches $r < r_g$. Though it may not appear so, $dr/d\tau$ is actually the velocity of the particle as measured by an observer at rest at radius r if $E = m$. For a temporal interval dt and radial interval dr in the spherical coordinates, the observer measures the temporal interval $dT = \sqrt{B}dt$ and radial interval $dR = dr/\sqrt{B}$ (which follows from the Schwarszchild metric) and so the velocity it measures is

$$\frac{dR}{dT} = \frac{1}{B}\frac{dr}{dt} = \frac{1}{B}\frac{dr}{d\tau}\frac{d\tau}{dt} = \frac{1}{B}\frac{dr}{d\tau}\left(\frac{m}{E}B\right) = \frac{dr}{d\tau} \tag{3.77}$$

where we have used the energy conservation equation $E = mB\,dt/d\tau$ in the process.

3.5 GRAVITATIONAL WAVES

In a way similar to electromagnetic waves, we expect curvature perturbations in General Relativity to propagate and have an independent existence from their sources. Naturally, we call such propagating curvature perturbations *gravitational waves*.

3.5.1 Electromagnetic waves

Take the Maxwell's equations $\partial_\mu F^{\mu\nu} = 0$ and plug in the plane-wave ansatz

$$A_\mu = a_\mu e^{ik_\nu x^\nu} \tag{3.78}$$

to get

$$\left(k_\gamma k^\gamma g_{\mu\nu} - k_\mu k_\nu\right) a^\nu = 0. \tag{3.79}$$

Suppose that the wave is propagating in the z-direction with $k_3 = k$, $k_1 = k_2 = 0$, $k_0 = \omega$. Setting $\mu = 1$ and $\mu = 2$, we get

$$\left(\omega^2 - k^2\right) a^1 = 0; \quad \left(\omega^2 - k^2\right) a^2 = 0 \tag{3.80}$$

and setting $\mu = 0$ or $\mu = 3$ we get the same equation

$$\omega a_3 - k a_0 = 0. \tag{3.81}$$

Interestingly, while there are 4 degrees of freedom in A^μ there are only 3 equations to constrain them, meaning that one of the degrees of freedom in A^μ is left unconstrained. This boils down to the gauge invariance in electrodynamics. Our formulation of electrodynamics is inherently redundant: configurations that are related to one another via a gauge transformation $A_\mu \to A_\mu - \partial_\mu \alpha$ are identical.

Let us use the gauge freedom to set $A_0 = 0$. If initially $A_0 \neq 0$, we can always perform a gauge transformation with $\alpha = A_0/(i\omega)$ to get rid of A_0. Then, it follows from (3.81) that $a_3 = 0$ and hence $A_3 = 0$. In the end, we have the following solutions

$$\begin{aligned} A_1 &= a_1 \exp -ik(z \pm t) \\ A_2 &= a_2 \exp -ik(z \pm t) \\ A_0 = A_3 &= 0. \end{aligned}$$

Thus, an electromagnetic wave is characterized by two amplitudes a_1 and a_2 (corresponding to two possible polarizations) and propagates at the speed of light.

3.5.2 Linearized gravity

Consider a small tensor perturbation $\delta g_{\mu\nu}$ in an otherwise flat spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu} \quad \delta g_{\mu\nu} \ll 1 \tag{3.82}$$

in empty space, where the Einstein's equation reads

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0.$$

As before, we use the ansatz $\delta g_{\mu\nu} = h_{\mu\nu} e^{ik_\gamma x^\gamma}$ with $k_3 = k$, $k_1 = k_2 = 0$, $k_0 = \omega$. Up to first order in $h_{\mu\nu}$, we have

$$R_{\mu\nu} = \frac{1}{2}\left(k^2 h_{\mu\nu} - k_\lambda k_\mu h^\lambda{}_\nu - k_\lambda k_\nu h^\lambda{}_\mu + k_\mu k_\nu h^\lambda{}_\lambda\right) \tag{3.83}$$

and

$$R = \frac{1}{2}\left(2k^2 h^\lambda{}_\lambda - 2k_\lambda k_\mu h^{\lambda\mu}\right). \tag{3.84}$$

Substituting these into the Einstein's equations, we get the following 10 equations

$$\begin{aligned}
00&: \quad (h_{11} + h_{22})k^2 = 0 \\
03&: \quad (h_{11} + h_{22})k\omega = 0 \\
02&: \quad (-h_{20}k + h_{23}\omega)k = 0 \\
01&: \quad (-h_{10}k + h_{13}\omega)k = 0 \\
33&: \quad (h_{11} + h_{22})\omega^2 = 0 \\
32&: \quad (-h_{20}k + h_{23}\omega)\omega = 0 \\
31&: \quad (-h_{10}k + h_{13}\omega)\omega = 0 \\
22&: \quad h_{00}k^2 + \omega(-2h_{30}k + h_{33}\omega) + h_{11}\left(\omega^2 - k^2\right) = 0 \\
21&: \quad h_{12}\left(\omega^2 - k^2\right) = 0 \\
11&: \quad h_{00}k^2 + \omega(-2h_{30}k + h_{33}\omega) + h_{22}\left(\omega^2 - k^2\right) = 0.
\end{aligned}$$

We can see that the triple $00, 03, 33$, the pair $02, 32$ and the pair $01, 31$ are the same equations, which leaves us with 6 different equations and 4 unconstrained degrees of freedom of $h_{\mu\nu}$

$$\begin{aligned}
\text{3 eqs. (00, 03, 33)}: &\quad h_{11} + h_{22} = 0 \\
\text{2 eqs. (02, 32)}: &\quad -h_{20}k + h_{23}\omega = 0 \\
\text{2 eqs. (01, 31)}: &\quad -h_{10}k + h_{13}\omega = 0 \\
\text{1 eq. (21)}: &\quad h_{12}(\omega^2 - k^2) = 0 \\
\text{1 eq. (22-11)}: &\quad (h_{11} - h_{22})(\omega^2 - k^2) = 0 \\
\text{1 eq. (22+11)}: &\quad h_{00}k^2 + \omega(-2h_{30}k + h_{33}\omega) = 0.
\end{aligned} \tag{3.85}$$

Analogous to what we have seen in electrodynamics, the 4 unconstrained degrees of freedom have to do with some gauge freedom. This time, it corresponds to the freedom to make coordinates transformations of the form

$$x^\mu \to x'^\mu = x^\mu + \zeta^\mu \tag{3.86}$$

where ζ^μ is small, which is worth 4 degrees of freedom. Under such a transformation, the metric transforms as $g'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} g^{\alpha\beta}$, or in terms of $h_{\mu\nu}$ and to linear order

$$\begin{aligned}
\eta^{\mu\nu} + h'^{\mu\nu} &= (\delta^\mu_\alpha + \partial_\alpha \zeta^\mu)(\delta^\nu_\beta + \partial_\beta \zeta^\nu)(\eta^{\alpha\beta} + h^{\alpha\beta}) \\
&\approx \eta^{\mu\nu} + h^{\mu\nu} + \partial_\alpha \zeta^\mu \delta^\nu_\beta \eta^{\alpha\beta} + \partial_\beta \zeta^\nu \delta^\mu_\alpha \eta^{\alpha\beta}.
\end{aligned} \tag{3.87}$$

Thus, $h_{\mu\nu}$ (the inverse of $h^{\mu\nu}$) transform as

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \zeta_\nu - \partial_\nu \zeta_\mu. \tag{3.88}$$

In particular, we have

$$\begin{aligned}
h_{00} &\to h_{00} - 2i\omega\zeta_0 \\
h_{01} &\to h_{01} - i\omega\zeta_1 \\
h_{02} &\to h_{02} - i\omega\zeta_2 \\
h_{03} &\to h_{03} - i\omega\zeta_3 - ik\zeta_0 \\
h_{11} &\to h_{11} \\
h_{12} &\to h_{12} \\
h_{13} &\to h_{13} - ik\zeta_1 \\
h_{22} &\to h_{22} \\
h_{23} &\to h_{23} - ik\zeta_2 \\
h_{33} &\to h_{33} - ik\zeta_3.
\end{aligned}$$

It turns out that h_{11}, h_{22}, h_{12} are coordinate invariants. We can use the freedom to do coordinate transformations to set h_{00}, h_{01}, h_{02}, and h_{33} to zero. Having done that, it is easy to see that 4 out of the 6 remaining Einstein equations are trivial

$$h_{13} = h_{23} = h_{30} = 0; \quad h_{11} + h_{22} = 0. \tag{3.89}$$

The other two are non-trivial

$$(\omega^2 - k^2)h_{12} = 0; \quad (h_{11} - h_{22})(\omega^2 - k^2) = 0. \tag{3.90}$$

Finally, the solution can be written as

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h & h_{12} & 0 \\ 0 & h_{12} & -h & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega t - ikz} \tag{3.91}$$

where $h = h_{11} = -h_{22}$ and $\omega = k$. (Don't confuse this h with the trace of $h_{\mu\nu}$!) Thus, a gravitational wave has 2 polarizations and propagates with the speed of light.

3.5.3 Gravitational radiation

Continuing the analogy with electrodynamics, an accelerating mass may radiate gravitational waves in a way similar to how an accelerated charge radiates electromagnetic waves. Consider a distribution of mass with spatial extent ℓ consisting of particles moving at non-relativistic velocities. This will be our source of gravitational waves. Suppose that there is an observer located at a distance $R \gg \ell$ away from the source trying to detect the gravitational waves radiated by the source. We assume that the perturbations created by the source are sufficiently weak that we can describe them by linearized gravity. In this case, the only non-trivial Einstein's equations are

$$\begin{cases} \frac{1}{2}\Box h_{12} & = 8\pi G T_{12} \\ \frac{1}{2}\Box h & = 8\pi G \frac{1}{2}(T_{22} - T_{11}). \end{cases} \tag{3.92}$$

We have encountered these types of equations repeatedly in electrodynamics. They can be solved with the method of Green's function. In the present case, the Green's function that we are looking for is one that satisfies

$$\Box_{x,t} G(x, t; x', t') = \delta^3(\mathbf{x} - \mathbf{x}')\delta(t - t') \tag{3.93}$$

namely the retarded Green's function

$$G(x, t; x', t') = \frac{\delta\left[t' - (t - |\mathbf{x} - \mathbf{x}'|)\right]}{4\pi|\mathbf{x} - \mathbf{x}'|}. \tag{3.94}$$

Hence, we can write the solution for h_{12} as

$$\begin{aligned} h_{12} &= 16\pi G \int T_{12}(\mathbf{x}', t')G(\mathbf{x}, t; \mathbf{x}', t') d^3x' dt' \\ &= 4G \int T_{12}(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|) \frac{1}{|\mathbf{x} - \mathbf{x}'|} d^3x' \\ &\approx \frac{4G}{R} \int T_{12}(\mathbf{x}', t - R) d^3x' \end{aligned} \tag{3.95}$$

where in the last step we have made used the assumptions that $\ell \ll R$ and that the particles making up the source are non-relativistic.

We will now derive the following identity

$$\boxed{\int T^{ij} d^3x = \frac{1}{2}\frac{\partial^2}{\partial t^2} \int T_{00} x^i x^j d^3x} \tag{3.96}$$

which will help us calculate the integral in (3.95). The starting point is the energy conservation equation $\partial_\nu T^{\mu\nu} = 0$. For $\mu = 0$ it yields

$$\partial_0 T^{00} + \partial_i T^{0i} = 0 \tag{3.97}$$

and for $\mu = i$ it yields

$$\partial_0 T^{i0} + \partial_j T^{ij} = 0. \tag{3.98}$$

Multiplying (3.97) with $x^i x^j$ and integrating, we get

$$\int \partial_0 T_{00} x^i x^j d^3x = -\int \partial_k T^{0k} x^i x^j d^3x. \tag{3.99}$$

After integrating by parts it becomes

$$\int \partial_0 T_{00} x^i x^j d^3x = \int \left(T^{0i} x^j + T^{0j} x^i\right) d^3x. \tag{3.100}$$

Then, take the time derivative ∂_0 of the above

$$\int \partial_0^2 T_{00} x^i x^j d^3x = \int \left(\partial_0 T^{0i} x^j + \partial_0 T^{0j} x^i\right) d^3x \tag{3.101}$$

and use (3.98) to rewrite it as

$$\int \partial_0^2 T_{00} x^i x^j \, d^3x = -\int \left(\partial_k T^{ik} x^j + \partial_k T^{jk} x^i \right) d^3x. \tag{3.102}$$

Integrating by parts once more bring us to what we are after, (3.96).

Therefore, using (3.96) we can recast (3.95) as

$$h_{12} \approx \frac{2G}{R} \int \frac{\partial^2}{\partial t^2} T_{00}(\mathbf{x}', t - R)\, x_1' x_2' \, d^3x'. \tag{3.103}$$

Defining the quadrupole moment of a mass distribution as

$$D^{ij} = \int \rho(x) \left[3x^i x^j - r^2 \delta^{ij} \right] dV \tag{3.104}$$

we can rewrite h_{12} as

$$h_{12} = \frac{2G}{3R} \ddot{D}_{12}. \tag{3.105}$$

Following similar steps for h, we get

$$h = \frac{2G}{3R} \frac{1}{2} \left(\ddot{D}_{22} - \ddot{D}_{11} \right). \tag{3.106}$$

How much energy do gravitational waves carry? Since $h_{\mu\nu}$ is analogous to A_μ in electrodynamics, we expect the energy density of gravitational waves to contain $\dot{h}^2$ or $\dot{h}_{12}^2$, whose dimensions are GeV^2. To get the right dimensions for energy density, GeV^4, we multiply it by G^{-1} so that the energy density ϵ is given by

$$\epsilon \propto G^{-1} \dot{h}^2. \tag{3.107}$$

This implies that a source radiating gravitational waves should lose energy at the rate

$$\frac{dE}{dt} \sim \int \epsilon \, dS \sim G^{-1} \left(\frac{G^2}{R^2} \dddot{D}^2 \right) R^2 \sim G \dddot{D}^2 \tag{3.108}$$

where we have used $h \sim G\ddot{D}/R$ (coming from (3.105) or(3.106)). Had we done the calculations properly, including all the numerical factors, we would get

$$\frac{dE}{dt} = -\frac{G}{45} \dddot{D}_{ij} \dddot{D}^{ij}. \tag{3.109}$$

3.5.4 Detecting gravitational waves

Consider two points A, B located on a plane orthogonal to the wavevector k_3 of a gravitational wave. The distance between these points is given by (here we only write the spatial part of the metric)

$$dl^2 = g_{ij} dx^i dx^j = \Delta x_1^2 + \Delta x_2^2 - h \left(\Delta x_1^2 - \Delta x_2^2 \right) - 2h_{12} \Delta x_1 \Delta x_2 \tag{3.110}$$

where h and h_{12} are defined in (3.91). As we can see, the metric perturbations $h_{\mu\nu}$ give rise to changes in the distance between two arbitrary points A and B. The term with h and the term with h_{12} give rise to two different types of deformations. The term with h squeezes and stretches the $x_1 x_2$ plane along the x_1 and x_2 directions (+ polarization). The term with h_{12} squeezes and stretches the $x_1 x_2$ plane along the $x_1 - x_2$ and $x_1 + x_2$ directions ($\times$ polarization). See Figure 4 for an illustration of the deforming effects of the two gravitational wave polarizations.

We can detect the said length changes using gravitational wave detectors such as LIGO (Laser Interferometer Gravitational waves Observatory). LIGO consists of two identical detectors—one in Hanford, Washington and one in Livingston,

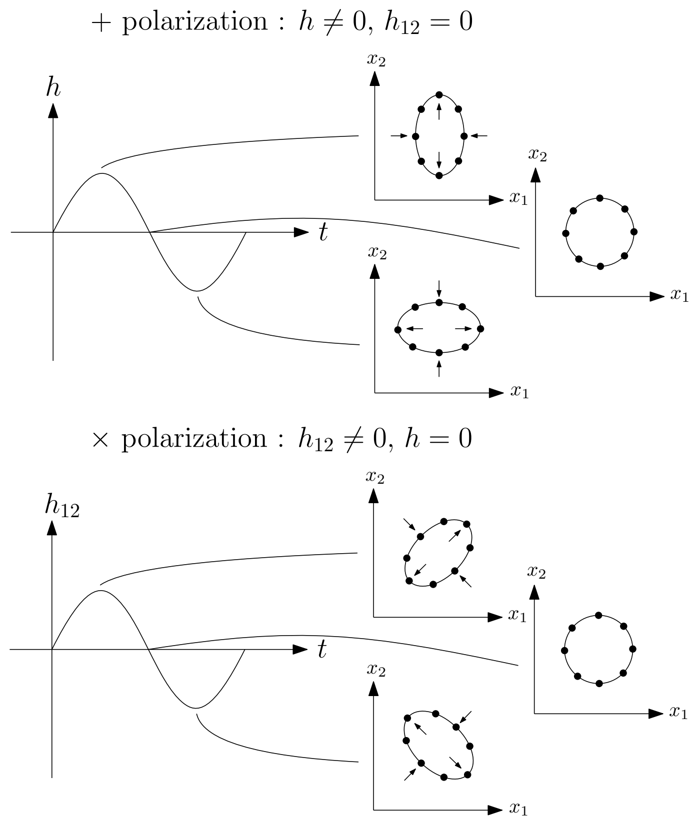

Figure 4: Two types of gravitational wave polarization and their (exaggerated) deforming effects on a set of points forming a circle in the x_1x_2 plane.

Louisiana—separated from each other by the distance light travels in 10 ms. Having two detectors at different locations gives us a better noise control (a true gravitational wave signal should show up nearly simultaneously in the two detectors in nearly identical forms) and allows us to triangulate the location of the gravitational wave source based on the time delay in the signals recorded in the two detectors. Each LIGO detector is a Michelson interferometer comprising of a laser, a beam splitter, 4 mirrors, and a photon detector, arranged in L shape with 4 km arm length as shown in Figure 5. A laser beam is directed toward a beam splitter which splits the beam into two, each going along one arm of the interferometer and reflected by the mirror placed at the end of it. Mirrors are also placed near the beam splitter so that each beam bounces back and forth along an arm hundreds of times before they arrive back at the beam splitter to be merged together again, making each arm effectively 1120 km long. In the absence of gravitational waves, the two beams would return to the beam splitter at the same time. A gravitational wave passing through the detector plane would in general have different lengthening/shortening effect on the two 4 km arms. As a result, light that has gone through the length of one arm multiple times would have accumulated a different phase than light that has gone through the length of the other arm the same number of times. The phase difference is seen as a variation in the intensity of light recorded by the photo detector.

Binary systems of massive and dense objects, e.g. a mutually orbiting pair of black holes, produce strong and unique gravitational wave signals that we can detect. Kip Thorne estimated that the typical length change caused by gravitational waves emitted by such systems is at the level of $\delta\ell/\ell \sim 10^{-21}$. If we take ℓ to be the arm length of a LIGO detector $\ell \sim 4$ km, the expected length change due to passing gravitational waves is around $\delta\ell \sim r_p/1000$, where r_p proton is the size. The task of measuring such a small length is made easier by repeatedly bouncing the light used in the interferometer along each arm.

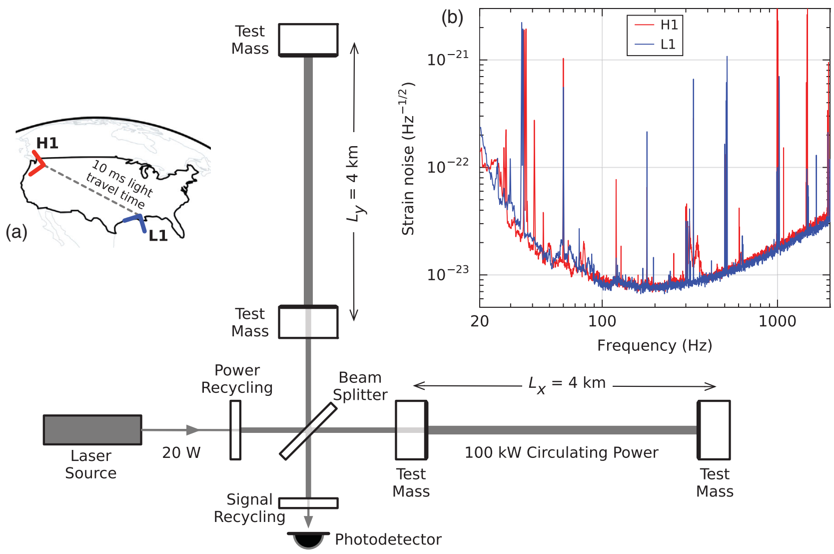

Figure 5: Schematic diagram of a LIGO detector. (a) Location and orientation of the LIGO detectors in Hanford, Washington (H1) and Livingston, Louisiana (L1) (b) The instrument noise of each detector near the time of the first gravitational wave detection.

On September 14, 2015 the two detectors of LIGO (nearly) simultaneously observed a transient gravitational-wave signal (see Figure 6). This event is dubbed GW150914 in reference to its date of occurrence. The signals match the waveform predicted by general relativity for a pair of black holes mutually orbiting and eventually coalescing into a single spinning black hole (see Figure 7). This observation provides us with the first direct evidence of the existence of gravitational waves, further supporting General Relativity as the theory of gravity.

Two years later, on August 17, 2017 a gravitational-wave signal named GW170817 was detected by both LIGO detectors and also by a third detector, VIRGO, located near Pisa, Italy. At the same time, strong gamma ray burst signals from the same direction, presumably from the same event, were observed by the Gamma-ray Burst Monitor on NASA's Fermi space telescope and the European Space Agency's gamma-ray observatory INTEGRAL. It is extremely unlikely that the simultaneous occurrence of all these signals is a chance coincidence. Thus, GW170817 gives an even more robust confirmation of the existence of gravitational waves than GW150914. An analysis of the combined data showed that the signal was consistent with a binary system of two objects in the mass range of neutron stars. Moreover, the non-detection of statistically-significant time delay between the arrival of gravitational and electromagnetic waves from the merging of the neutron stars puts a strong constraint on the deviation of the propagation speed of gravitational wave from the speed of light, i.e. yet another confirmation of General Relativity which predicts that gravitational waves should travel at the speed of light.

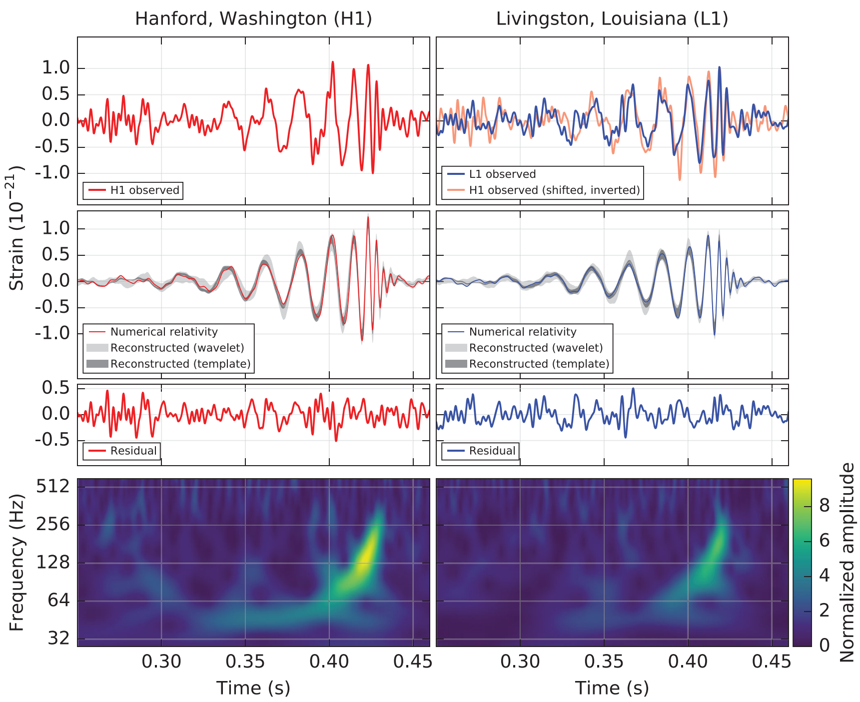

Figure 6: The gravitational-wave event GW150914 observed by the two LIGO detectors

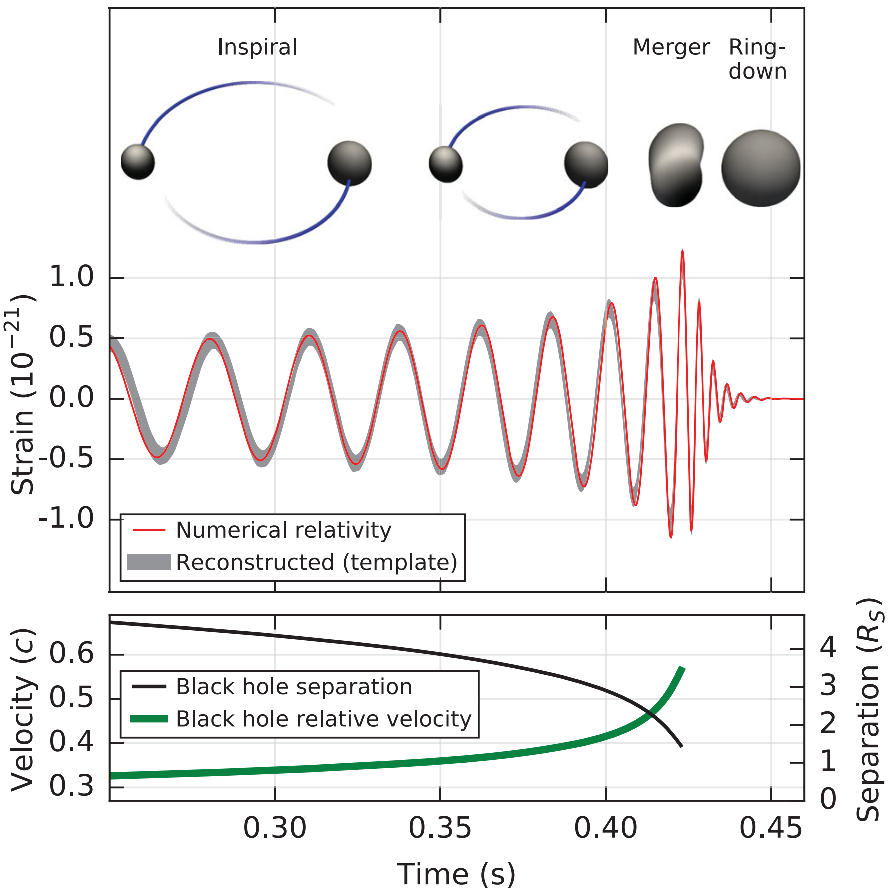

Figure 7: Top: Comparison between the gravitational-wave strain amplitude as a function of time of the GW150914 event observed by LIGO and predicted with numerical general relativity models. Bottom: The effective relative velocity and relative separation of the two black holes as functions of time.