RELATIVITY AND COSMOLOGY II

Problem Set 7 1st April 2025

1. Blackbody radiation

Suppose that at $T \approx 0.25$ eV the photons followed the black-body distribution

$$dE_\omega = \frac{V\hbar}{\pi^2 c^3}\frac{\omega^3 d\omega}{e^{\hbar\omega/kT}-1} \ , \tag{1}$$

and after that they stopped interacting. Show that the above distribution keeps its shape when the universe expands. Deduce the temperature dependence as a function of scale factor and check that the total entropy $S = V \cdot s = V \cdot \frac{2\pi^2}{45}T^3$ is conserved.

2. Photon decoupling in numbers

The temperature of the Cosmic Microwave Background today is $T_{CMB} = 2.73$ K. Assuming that the photon decoupling took place at $T_d \approx 0.25$ eV, estimate

1. The redshift of decoupling z_d,
2. The age of the Universe at this redshift,
3. The abundances $\Omega_\Lambda,\ \Omega_m,\ \Omega_\gamma$ at z_d.

According to the ΛCDM model, the abundances today are

$$\Omega_\Lambda \approx 0.69 \pm 0.01, \quad \Omega_m \approx 0.31 \pm 0.01, \quad \Omega_\gamma \approx 8.2 \times 10^{-5} \ .$$

3. Fermion number density

Consider an equilibrium cosmic plasma composed of fermions and antifermions of the same species with the mass m and the chemical potential μ for particles (and $-\mu$ for antiparticles). Show that in the limit $m \ll T$ and $\mu \ll T$ the difference between their number densities is given by

$$\Delta n = n_f - n_{\bar{f}} \approx \mu g \frac{T^2}{6}.$$

Indication: You may need the formula $\int_0^\infty \frac{z}{e^z+1}dz = \frac{\pi^2}{12}$.