RELATIVITY AND COSMOLOGY II

Problem Set 6 26th March 2024

1. Densities evolution

The distribution function $n(t, p^i)$ of (non-interacting) particles in an expanding Universe obeys

$$\frac{\partial n}{\partial t} - Hp^i \frac{\partial n}{\partial p^i} = 0 \ ,$$

where H is the Hubble parameter, $i = x, y, z$, and summation over repeated indices is understood.

1. Show that, by expanding the Hubble parameter $H = \frac{\dot{a}}{a}$ one can rewrite the equation as

$$a \frac{\partial n}{\partial a} - p^i \frac{\partial n}{\partial p^i} = 0 \ . \tag{1}$$

2. Prove that, along the curve $a(s), p^i(s)$ the value of the distribution function $n(a(s), p^i(s))$ is constant if the curve follows the equation

$$\frac{da}{ds} = a \ , \qquad \frac{dp^i}{ds} = -p^i \ .$$

3. Find explicit expression for this curve (called the characteristics curve).

4. Show that time evolution of the distribution function simply follow the redshift – i.e. show using characteristic curves that $p^i(t) = p^i(t_0)\frac{a(t_0)}{a(t)}$.

2. Conservation of chemical potential

From the lecture notes we know that the Boltzmann equation is of the form of the balance equation, whose r.h.s. (the collision integral $I_{col.}$) measures the difference between gain and loss processes in a given cell of the phase space. In equilibrium, the gain and loss processes balance each other and $I_{col.} = 0$.

1. Consider the $2 \to 2$ reaction, $(1) + (2) \to (3) + (4)$. Assign a chemical potential to each particle participating in the reaction and prove that in equilibrium

$$\mu_1 + \mu_2 = \mu_3 + \mu_4 \ . \tag{2}$$

Indication: Use that in equilibrium the distribution function f_i of a particle i is known. It may be useful to compute the quantity $f_i/(1 \pm f_i)$.

Note: The equation (2) always has a trivial solution, $\mu_i = 0$ for all i. In fact, a non-zero chemical potential can only be assigned to a particle provided that the particle carries some quantum numbers (an electric charge, baryon and lepton numbers, etc.) that are conserved in the reactions.

2. Can you make a guess how the result (2) generalizes to a multiparticle reaction $n \to m$, that is $(1) + ... + (n) \to (\bar{1}) + ... + (\bar{m})$? (No computation is needed here.)

3. Show that the chemical potential of a photon is zero, $\mu_\gamma = 0$.

   Indication: For this and the next question, you should consider an exemplary reaction.

4. Show that the chemical potentials of particle (p) and antiparticle ($\bar{p}$) are related by $\mu_p = -\mu_{\bar{p}}$.

3. Proton gas

Consider a universe dominated by radiation containing photons (γ), protons (p), antiprotons ($\bar{p}$) and pions (π). Consider the reaction $p\bar{p} \leftrightarrow n\pi\ , n \geq 2$, whose cross section is $\sigma = m_\pi^{-2}$, where m_π is the pion's mass. Let also $n_p = n_{\bar{p}}$.

1. Find the decoupling temperature T_d assuming that protons are non-relativistic.

2. Find the ratio between proton and photon number densities at the time of decoupling.

3. Compute the number density of photons in the universe at the time when the temperature is $T = 2.73K$. Estimate the number density of protons at that time.